# CSE 311 Lecture 28: Undecidability of the Halting Problem

Emina Torlak and Kevin Zatloukal

# Topics

**Final exam**

Logistics, format, and topics.

**Countability and uncomputability**

A quick recap of Lecture 27.

**Undecidability of the halting problem**

Important problems computers can’t solve.

# Final exam

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**Monday, December 10 in JHN 102**

Section A at 16:30-18:20,

Section B at 14:30-16:20.

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**Important: take the exam at the time assigned to your section.**

If you have a scheduling conflict, email the staff ASAP.

Bring your UW ID and have it ready to be checked during the exam.

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**Final review session is on Sunday, Dec 09 at 15:00-17:00 in GWN 301.**

Bring your questions!

# Final exam format and topics

## Format

**8 problems** in **110 minutes**.

Closed book, closed notes, no calculators, no cellphones.

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**Questions and topics**

(1) Write a regex, DFA, CFG for given languages.
(2) A formal proof.
(3) A proof by strong induction.
(4) A proof by structural induction.
(5) NFA to DFA conversion.
(6) DFA minimization.
(7) A proof that a language is irregular.
(8) Short answers on modular arithmetic, relations, logic, uncomputability.

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You've solved similar problems on homeworks and in sections.

**Do the easy parts of all the problems first.** Don't get stuck on one problem!

# Countability and uncomputability

A quick recap of Lecture 27.

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- $\Sigma^*$ over finite $\Sigma$
- All (Java) programs

Shown by dovetailing.

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**Uncountable sets**

- All real numbers in $[0, 1)$
- All functions from $\mathbb{N}$ to $\{0, 1\}$

Shown by diagonalization.

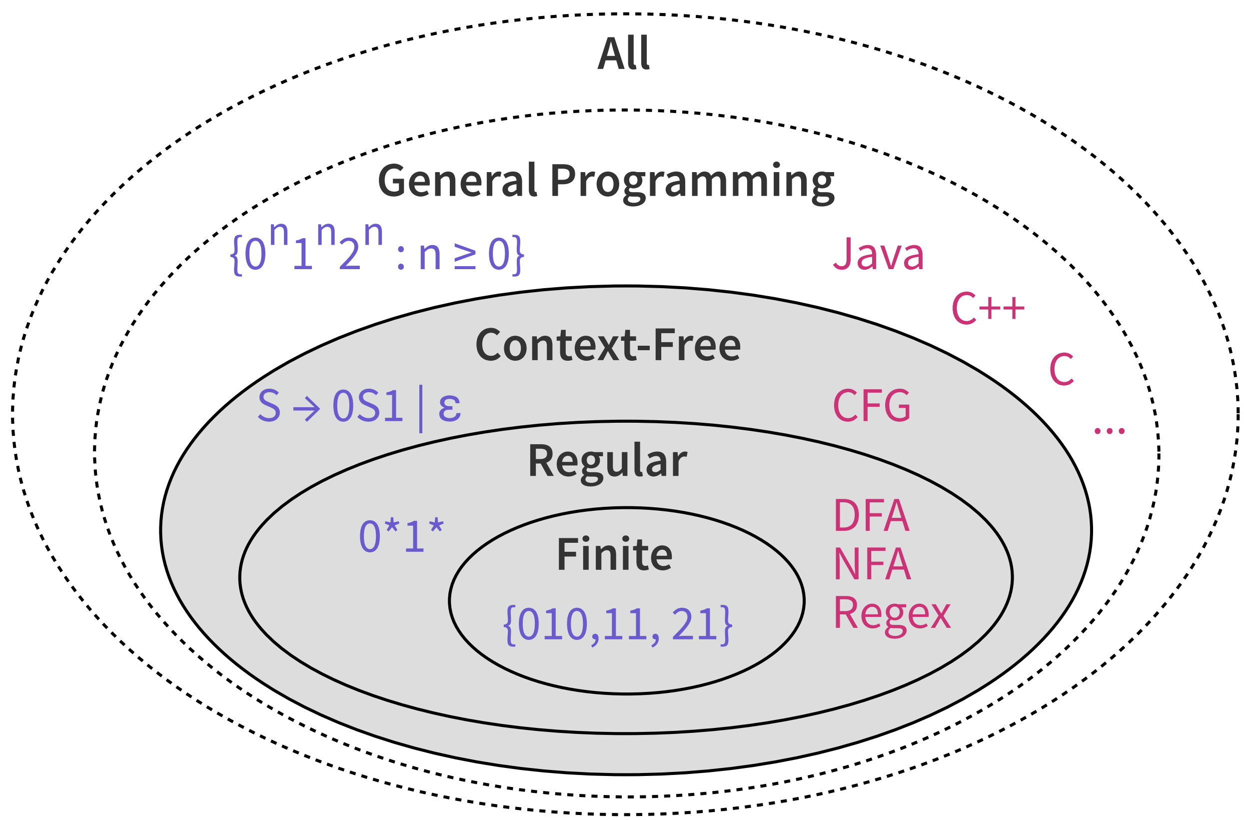
# Uncomputable functions

**We have seen that ...**

The set of all (Java) programs is countable.

The set of all functions $f : \mathbb{N} \rightarrow \{0, 1\}$ is uncountable.

So there must be some function that is not computable by any program!



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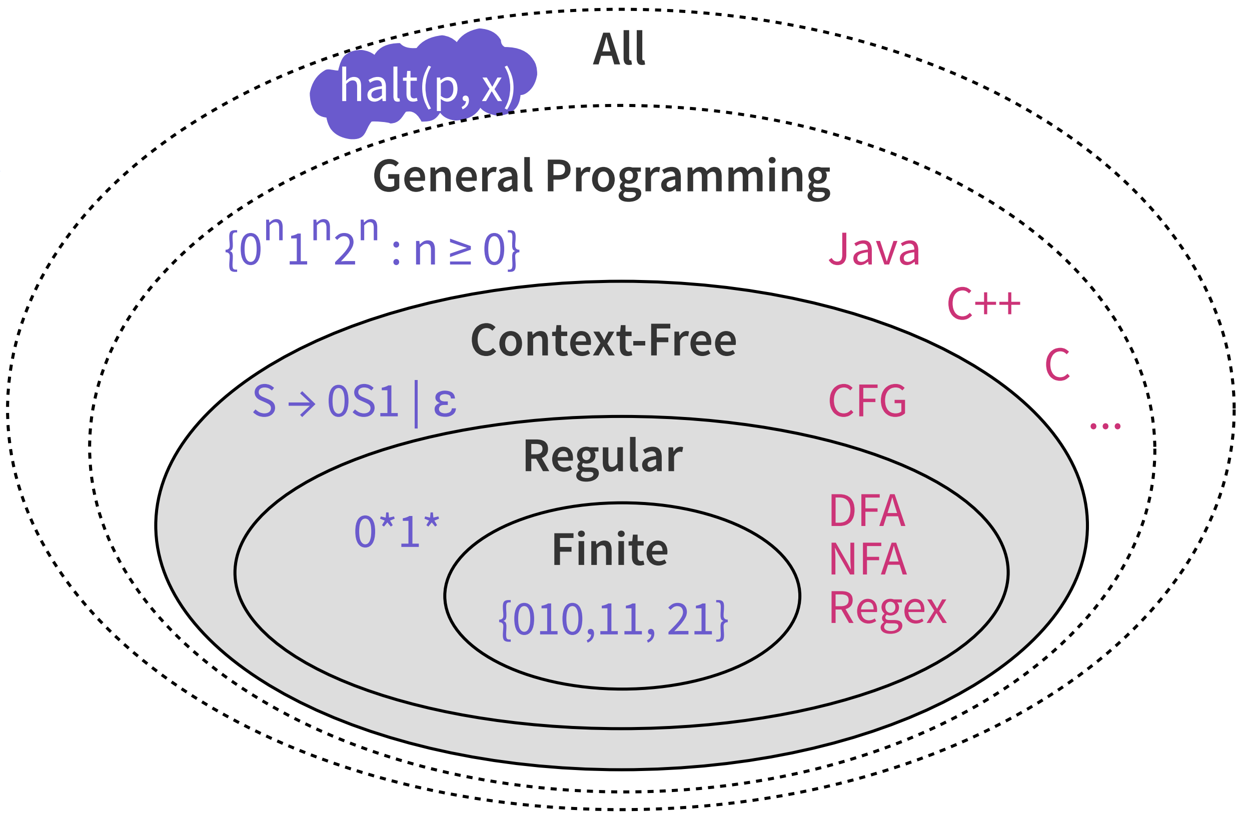
We'll study one such important function today.



# Undecidability of the halting problem

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**What is `P(code(P))`?**

true

# And now, the halting problem!

**The Halting Problem**

Given an input $x$ and `code(P)` for any program $P$,
output `true` if $P$ halts on the input $x$, and
output `false` if $P$ does not halt (diverges) on the input $x$.

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**Can't we determine this by just running $\mathbf{P}$ on $\mathbf{x}$?**

No! We can't tell if $P$ diverged on $x$ or is taking a long time to return.

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**Theorem (due to Alan Turing)**

There is no program that solves the halting problem.

In other words, there is no program that computes the function described by the halting problem. This function is therefore uncomputable. Because the function outputs a boolean (a yes/no decision), we say that the underlying problem is *undecidable*.

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Then, we can write the program `D` as follows:

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**So we reach a contradiction in either case.**

Therefore, our assumption that `H` exists must be false. □

# Where did the idea for creating `D` come from?

```
public static void D(Object x) {
  if (H(x, x) == true) {
    while (true); // diverge
  } else {
    return;       // halt
  }
}
```

**Note that `D` halts on `code(P)`**

iff `H(code(P),code(P))` outputs false, i.e.,

iff `P` doesn't halt on the input `code(P)`.

**Therefore, `D` differs from every program `P` on the input `code(P)`.**

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This sounds like diagonalization!

# “D” is for diagonalization

**List all Java programs.**

This list exists because the set of all Java programs is countable.

| | |
|---|---|
| $P_0$ | |
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| $P_2$ | |
| $P_3$ | |
| $P_4$ | |

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|---|---|---|---|---|---|---|
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**Let $\langle \mathbf{P} \rangle$ stand for `code(P)`.**

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**`D` behaves like the flipped diagonal**

$D(\langle P \rangle) = \neg P(\langle P \rangle)$, and differs from every $P$ in the list.

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**`D` behaves like the flipped diagonal**

$D(\langle P \rangle) = \neg P(\langle P \rangle)$, and differs from every $P$ in the list.

**But the list is complete.**

So if `D` isn't included, it cannot exist!

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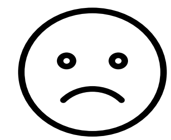

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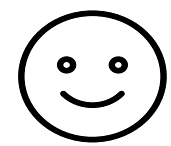
**But we can often decide these questions in practice!**



They are undecidable for *arbitrary* programs and properties.

Yet decidable for many specific classes of programs and properties.

And when we allow "yes/no/don't know" answers.

# That's all folks!

Propositional logic.
Boolean logic, circuits, and algebra.
Predicates, quantifiers and predicate logic.
Inference rules and formal proofs for propositional and predicate logic.
English proofs.
Set theory.
Modular arithmetic and prime numbers.
GCD, Euclid's algorithm, modular inverse, and exponentiation.
Induction and strong induction.
Recursively defined functions and sets.
Structural induction.
Regular expressions.
Context-free grammars and languages.
Relations, composition, and reflexive-transitive closure.
DFAs, NFAs, and product construction for DFAs.
Finite state machines with output.
Minimization algorithm for finite state machines.
Conversion of regular expressions to NFAs.
Subset construction to convert NFAs to DFAs.
Equivalence of DFAs, NFAs, regular expressions.
Method to prove languages are not regular.
Cardinality, countability, and diagonalization.
Undecidability and the halting problem.

Go forth and prove great things!