CSE 311 Lecture 27: Cardinality and Uncomputability

Emina Torlak and Kevin Zatloukal

Topics

Course evaluation

Is open; please tell us what you think!

Languages and representations

How powerful are general-purpose programming languages?

Cardinality and countability

What does it mean for two sets to have the same size?

Uncomputability

Are there problems computers can’t solve?

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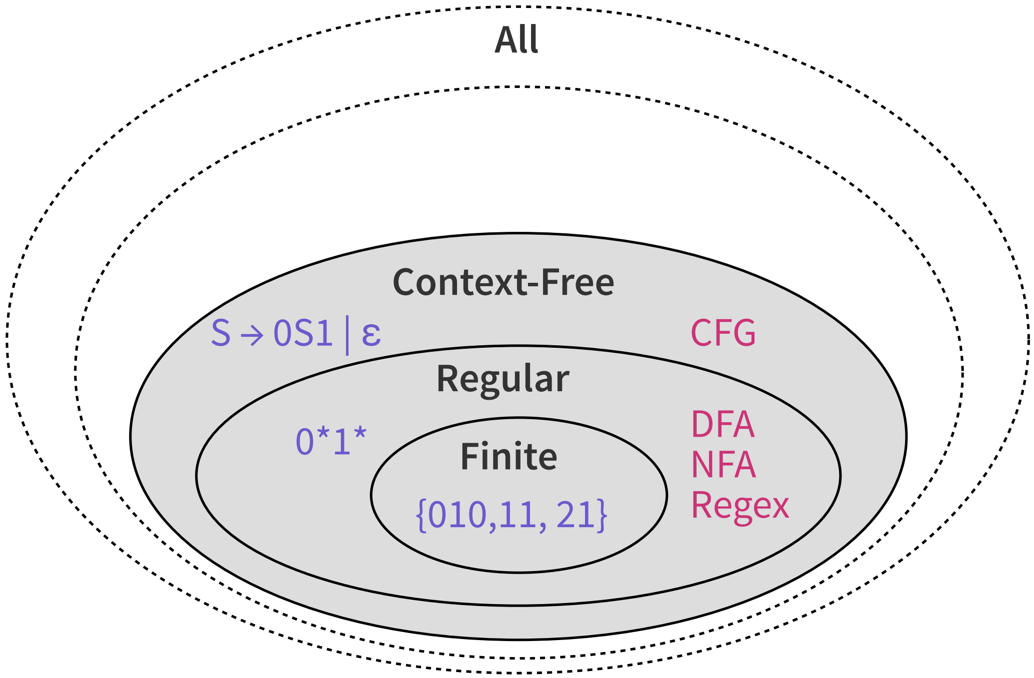
Languages and representations

How powerful are general-purpose programming languages?

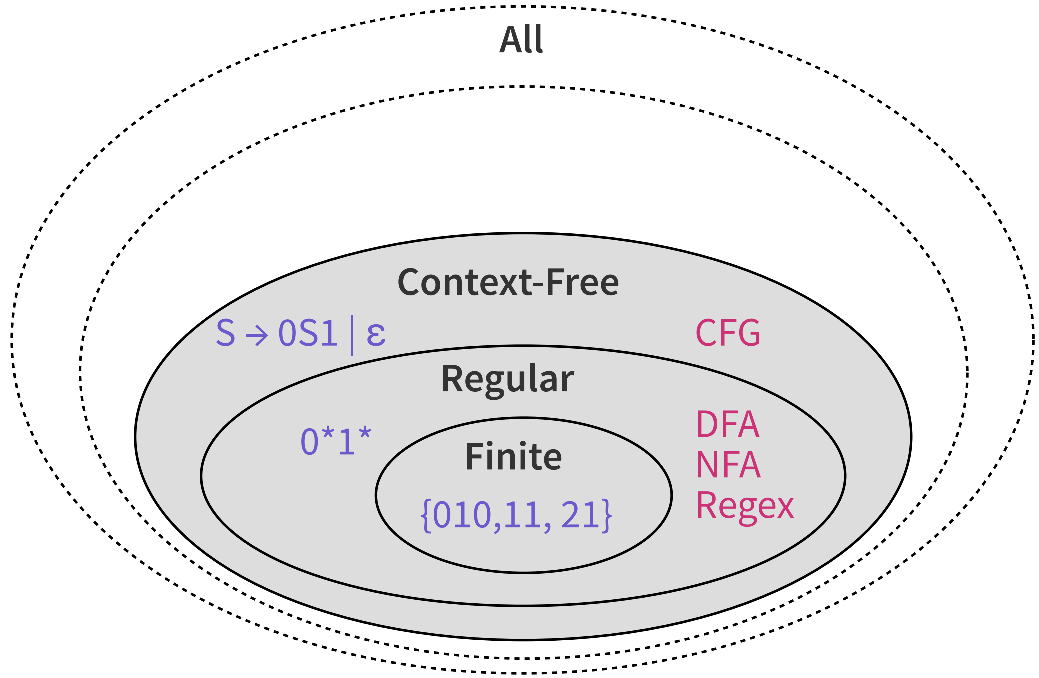
A hierarchy of languages and representations



A hierarchy of languages and representations

We can think of languages as functions from strings to booleans.

Such a function returns `true` iff a string is in the language.



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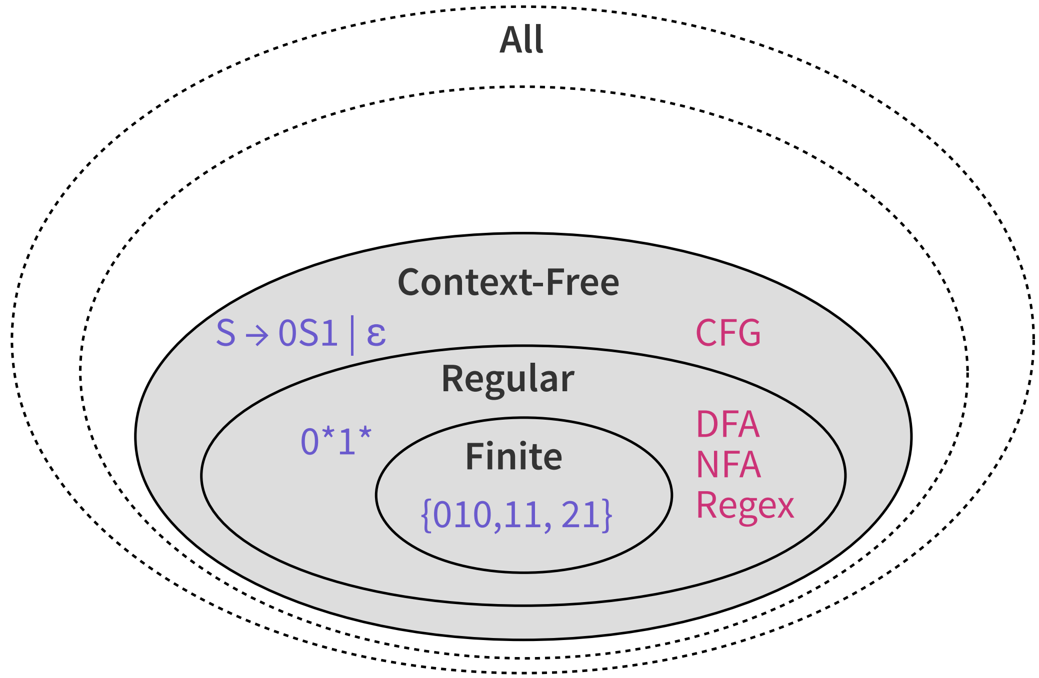

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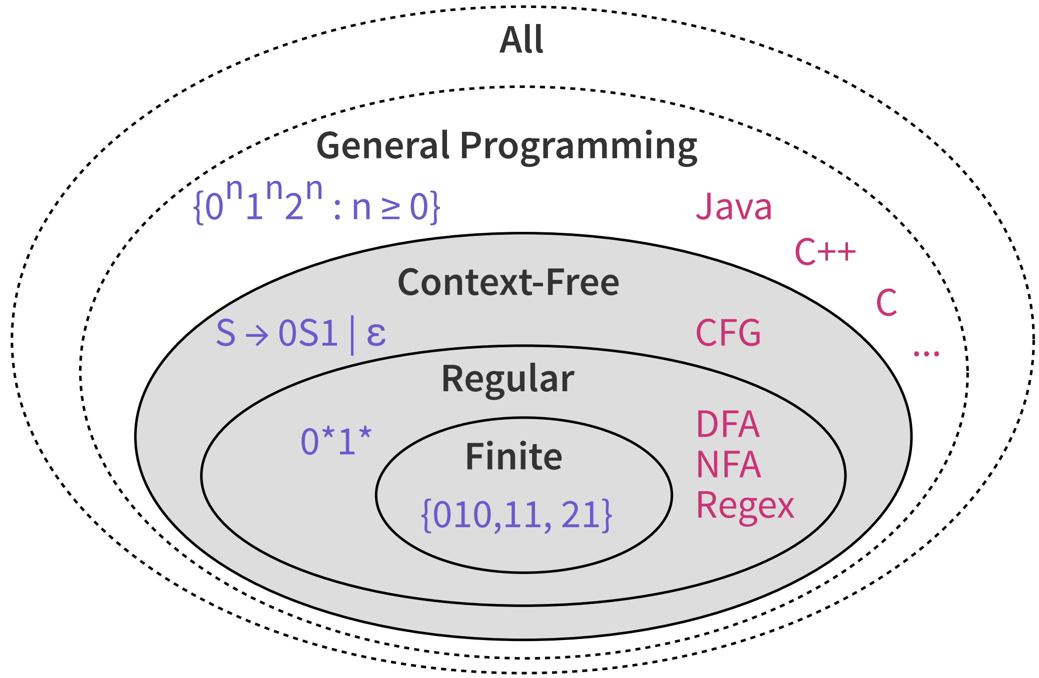

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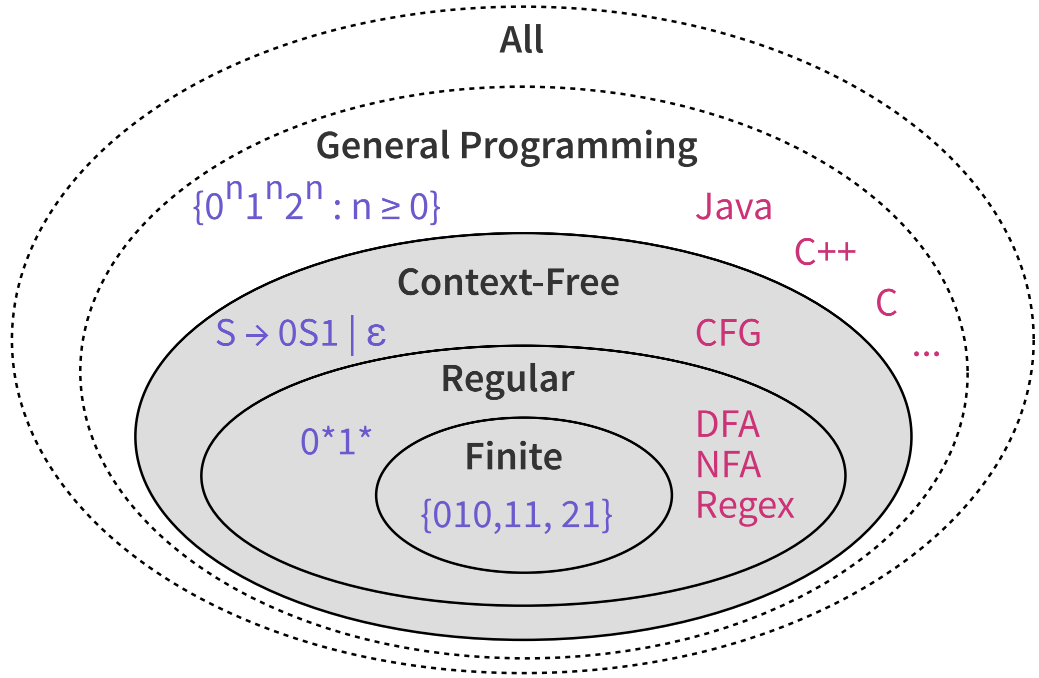
Are there some functions no program can represent?

That's what we'll study in these last two lectures :)



Cardinality and countability

What does it mean for two sets to have the same size?

Understanding cardinality

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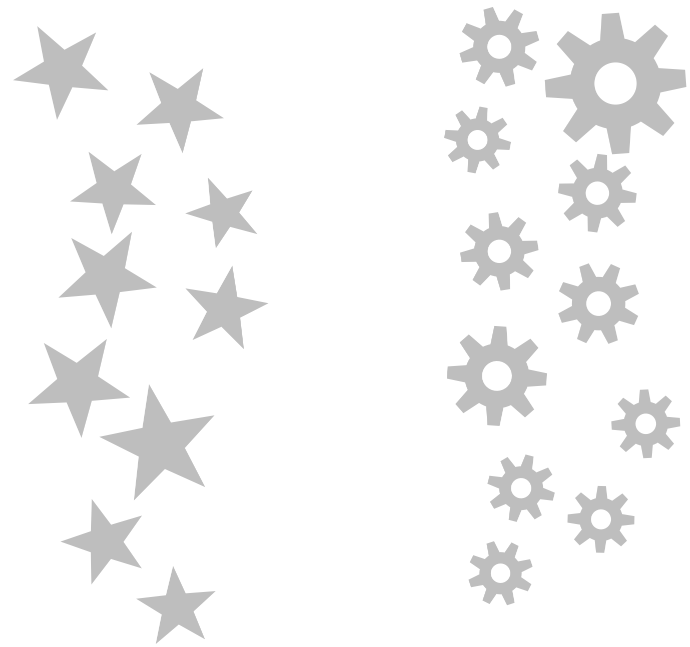
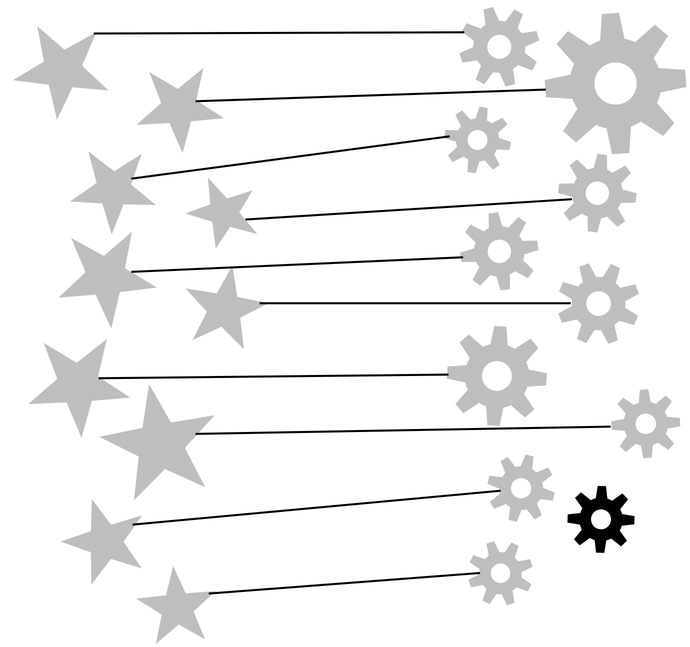


Understanding cardinality

What does it mean for two sets to have the same size?

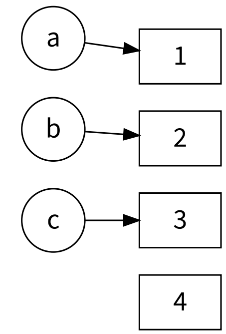
We can establish a *one-to-one correspondence* between their elements.



Defining one-to-one correspondences

One-to-one (injective) functions

A function $f : A \to B$ is *one-to-one* (1-1) if every output corresponds to at most one input: $f(x) = f(x') \Rightarrow x = x'$ for all $x, x' \in A$.



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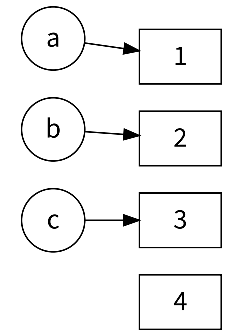
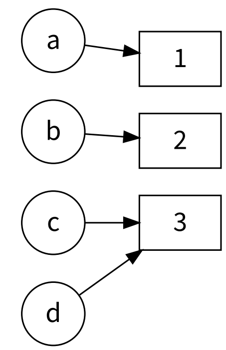


Onto (surjective) functions

A function $f : A \to B$ is *onto* if there is at least one input for every output: for each $y \in B$, there is an $x \in A$ such that $f(x) = y$.



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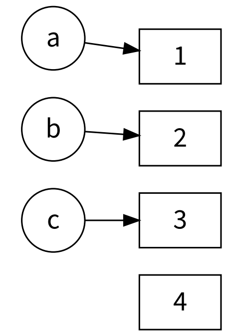

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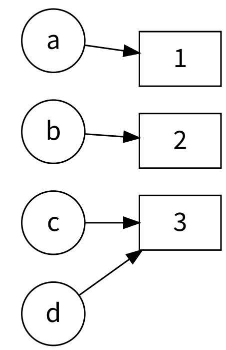
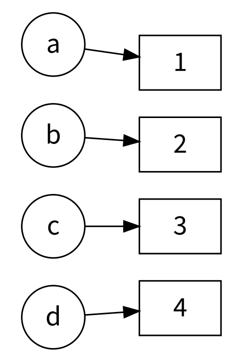


One-to-one correspondences (bijections)

A function $f : A \to B$ is a *one-to-one correspondence* if it is both one-to-one and onto.



Defining cardinality

Cardinality of two sets

Sets A and B have the same *cardinality* if there is a one-to-one correspondence between them, i.e., there is a bijection $f : A \to B$.

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Example: do $\mathbb{N}$ and even natural numbers have the same cardinality?

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Sets A and B have the same *cardinality* if there is a one-to-one correspondence between them, i.e., there is a bijection $f : A \to B$.

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Example: do $\mathbb{N}$ and even natural numbers have the same cardinality?

Yes! The 1-1 correspondence is $f(n) = 2n$.

0	1	2	3	4	5	6	7	8	9	10	...
0	2	4	6	8	10	12	14	16	18	20	...

Countable sets

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A set S is countable iff we can order its elements: $S = \{x_0, x_1, x_2, \ldots\}$.

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Example: is the set $\mathbb{Z}$ of all integers countable?

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	9	10	...
$\mathbb{Z}$	0	1	-1	2	-2	3	-3	4	-4	5	-5	...

Is the set $\mathbb{Q}^+$ of positive rationals countable?

There are infinitely many rationals between any two rational numbers.

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1 / 1	1 / 2	1 / 3	1 / 4	1 / 5	1 / 6	1 / 7	1 / 8	…
2 / 1	2 / 2	2 / 3	2 / 4	2 / 5	2 / 6	2 / 7	2 / 8	…
3 / 1	3 / 2	3 / 3	3 / 4	3 / 5	3 / 6	3 / 7	3 / 8	…
4 / 1	4 / 2	4 / 3	4 / 4	4 / 5	4 / 6	4 / 7	4 / 8	…
5 / 1	5 / 2	5 / 3	5 / 4	5 / 5	5 / 6	5 / 7	5 / 8	…
6 / 1	6 / 2	6 / 3	6 / 4	6 / 5	6 / 6	6 / 7	6 / 8	…
7 / 1	7 / 2	7 / 3	7 / 4	7 / 5	7 / 6	7 / 7	7 / 8	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	…

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4 / 1	4 / 2	4 / 3	4 / 4	4 / 5	4 / 6	4 / 7	4 / 8	...
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7 / 1	7 / 2	7 / 3	7 / 4	7 / 5	7 / 6	7 / 7	7 / 8	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	...

Counting $\mathbb{Q}^+$ with dovetailing

The set of all positive rational numbers is countable.

$$\mathbb{Q}^+ = \{1/1, 2/1, 1/2, 3/1, 2/2, 1/3, 4/1, 2/3, 3/2, \ldots\}$$

List elements in the order of the sum of the numerator and denominator, breaking ties according to the denominator.

Only k pairs of positive numbers add up to $k+1$, so every positive rational number comes up some point.

This technique is called *dovetailing*.

Σ^* is countable for every finite Σ

How would we show this?

Alphabetical / lexicographic order doesn't work (infinitely many A's):

A, AA, AAA, AAAA, AAAAA, . . .

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Use dovetailing again!

List strings in the order of length, breaking ties lexicographically.

There are only $|\Sigma|^k$ strings on length k.

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Use dovetailing again!

List strings in the order of length, breaking ties lexicographically.

There are only $|\Sigma|^k$ strings on length k.

For example, $\{\mathbf{0}, \mathbf{1}\}^*$ is countable:

$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, \ldots\}$

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So, is everything countable?

Real numbers are not countable

Theorem [due to Cantor]

The set of real numbers between 0 and 1, $[0, 1)$, is not countable.

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Theorem [due to Cantor]

The set of real numbers between 0 and 1, $[0, 1)$, is not countable.

Proof will be by contradiction. Using a method called *diagonalization*.

Proof that $[0, 1)$ is uncountable: preliminaries

First, note that every number in $[0, 1)$ has an infinite decimal expansion:

$$
\begin{aligned}
1/2 &= 0.50000000000000000000000\ldots \\
1/3 &= 0.33333333333333333333333\ldots \\
1/7 &= 0.14285714285714285714285\ldots \\
\pi - 3 &= 0.14159265358979323846264\ldots \\
1/5 &= 0.19999999999999999999999\ldots \\
&= 0.20000000000000000000000\ldots
\end{aligned}
$$

This representation is unique except for the cases where the decimal expansion ends in all 0's or all 9's. We will use the all 0's representation.

Proof that $[0, 1)$ is uncountable: diagonalization

Suppose for contradiction that there is a list $\{r_0, r_1, r_2, \ldots\}$ of all real numbers in $[0, 1)$.

$$
\begin{array}{ll}
r_0 & 0.500000000000\ldots \\
r_1 & 0.333333333333\ldots \\
r_2 & 0.142857142857\ldots \\
r_3 & 0.141592653589\ldots \\
r_4 & 0.200000000000\ldots \\
\vdots &
\end{array}
$$

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Suppose for contradiction that there is a list $\{r_0, r_1, r_2, \ldots\}$ of all real numbers in $[0, 1)$.

Consider the digits $x_0, x_1, x_2, x_3, \ldots$ on the diagonal of this list, i.e., the n-th digit of r_n for $n \in \mathbb{N}$.

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For each such digit x_i, construct the digit $\widehat{x}_i$ as follows:

- If $x_i = 1$ then $\widehat{x}_i = 0$.
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Now, consider the number $\hat{r} = 0.\hat{x}_0 \hat{x}_1 \hat{x}_2 \hat{x}_3 \ldots$

r_0	$0.500000000000\ldots$
r_1	$0.333333333333\ldots$
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Note that $r_n \neq \widehat{r}$ for any $n \in \mathbb{N}$ because they differ on the n-th digit.

r_0	$0.\mathbf{5}00000000000 \ldots$
r_1	$0.3\mathbf{3}3333333333 \ldots$
r_2	$0.14\mathbf{2}857142857 \ldots$
r_3	$0.141\mathbf{5}92653589 \ldots$
r_4	$0.2000\mathbf{0}0000000 \ldots$
$\vdots$	

Proof that $[0, 1)$ is uncountable: diagonalization

Suppose for contradiction that there is a list $\{r_0, r_1, r_2, \dots\}$ of all real numbers in $[0, 1)$.

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So the list doesn't include $\widehat{r}$, which is a contradiction. Thus the set $[0, 1)$ is uncountable.

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\begin{array}{ll}
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Suppose for contradiction that there is a list $\{f_1, f_2, f_3, \ldots\}$ of functions from $\mathbb{N}$ to $\{0, 1\}$.

f_0	0 0 0 0 0 0 0 0 0 …
f_1	1 1 1 1 1 1 1 1 1 …
f_2	0 1 0 1 0 1 0 1 0 …
f_3	0 1 1 1 0 1 1 1 0 …
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Consider the outputs $x_0, x_1, x_2, x_3, \ldots$ on the diagonal of this list, i.e., $f_n(n)$ for $n \in \mathbb{N}$.

f_0	0	0	0	0	0	0	0	0	0	…
f_1	1	1	1	1	1	1	1	1	1	…
f_2	0	1	0	1	0	1	0	1	0	…
f_3	0	1	1	1	0	1	1	1	0	…
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Now, consider the function $\widehat{f}(n) = \widehat{x}_n$.

Note that $f_n \neq \widehat{f}$ for any $n \in \mathbb{N}$ because the functions differ on the n-th output.

So the list doesn't include $\widehat{f}$, which is a contradiction. Thus the set $\{f \mid f : \mathbb{N} \to \{0, 1\}\}$ is uncountable.

f_0	0 0 0 0 0 0 0 0 0 …
f_1	1 1 1 1 1 1 1 1 1 …
f_2	0 1 0 1 0 1 0 1 0 …
f_3	0 1 1 1 0 1 1 1 0 …
f_4	1 1 0 0 0 1 1 0 0 …
⋮	

Uncomputability

Are there problems computers can't solve?

Uncomputable functions

We have seen that …

The set of all (Java) programs is countable.

The set of all functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable.

Uncomputable functions

We have seen that ...

The set of all (Java) programs is countable.

The set of all functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable.

So there must be some function $f : \mathbb{N} \to \{0, 1\}$ that is not computable by any program! We'll study one such function next time.

Summary

Cardinality and countability.

Two sets have the same cardinality if there is a bijection between them.
A set is countable iff it has the same cardinality as some subset of $\mathbb{N}$.
Use dovetailing to show that a set is countable and diagonalization to show that it's uncountable.

Computability.

Countability of programs and uncountability of functions $f : \mathbb{N} \to \{0, 1\}$ tells us that there is some function that can't be computed by any program!