Topics

Homework 6 advice
  Start early!

Recursively defined sets
  Recursive definitions of sets.

Structural induction
  A method for proving properties of recursive structures.

Using structural induction
  Example proofs about recursively defined numbers, strings, and trees.
Homework 6 advice

Start early!
Homework 6 isn’t necessarily harder …

But you may find it to be more work than most other assignments. So please start early :)

Pay special attention to Problem 6.4.

Requires keeping careful track of
(1) what you know and
(2) what you need to prove.
Recursively defined sets

Recursive definitions of sets.
Giving a recursive definition of a set

A recursive definition of a set $S$ has the following parts:

- **Basis step** specifies one or more initial members of $S$.
- **Recursive step** specifies the rule(s) for constructing new elements of $S$ from the existing elements.
- **Exclusion (or closure) rule** states that every element in $S$ follows from the basis step and a finite number of recursive steps.
Giving a recursive definition of a set

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- **Exclusion (or closure) rule** states that every element in $S$ follows from the basis step and a finite number of recursive steps.

The exclusion rule is assumed, so no need to state it explicitly.
Examples of recursively defined sets

Natural numbers
   Basis: \( 0 \in S \)
   Recursive: if \( n \in S \), then \( n + 1 \in S \)
Examples of recursively defined sets

Natural numbers
  Basis: $0 \in S$
  Recursive: if $n \in S$, then $n + 1 \in S$

Even natural numbers
Examples of recursively defined sets

Natural numbers
   Basis: \( 0 \in S \)
   Recursive: if \( n \in S \), then \( n + 1 \in S \)

Even natural numbers
   Basis: \( 0 \in S \)
Examples of recursively defined sets

Natural numbers
  Basis: $0 \in S$
  Recursive: if $n \in S$, then $n + 1 \in S$

Even natural numbers
  Basis: $0 \in S$
  Recursive: if $x \in S$, then $x + 2 \in S$
Examples of recursively defined sets

Natural numbers
   Basis: $0 \in S$
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   Basis: $0 \in S$
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Powers of 3
Examples of recursively defined sets

Natural numbers
Basis: $0 \in S$
Recursive: if $n \in S$, then $n + 1 \in S$

Even natural numbers
Basis: $0 \in S$
Recursive: if $x \in S$, then $x + 2 \in S$

Powers of 3
Basis: $1 \in S$
Examples of recursively defined sets

Natural numbers
   Basis: $0 \in S$
   Recursive: if $n \in S$, then $n + 1 \in S$

Even natural numbers
   Basis: $0 \in S$
   Recursive: if $x \in S$, then $x + 2 \in S$

Powers of 3
   Basis: $1 \in S$
   Recursive: if $x \in S$, then $3x \in S$
Examples of recursively defined sets

Natural numbers
   Basis: \(0 \in S\)
   Recursive: if \(n \in S\), then \(n + 1 \in S\)

Even natural numbers
   Basis: \(0 \in S\)
   Recursive: if \(x \in S\), then \(x + 2 \in S\)

Powers of 3
   Basis: \(1 \in S\)
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Fibonacci numbers
Examples of recursively defined sets

Natural numbers
Basis: $0 \in S$
Recursive: if $n \in S$, then $n + 1 \in S$

Even natural numbers
Basis: $0 \in S$
Recursive: if $x \in S$, then $x + 2 \in S$

Powers of 3
Basis: $1 \in S$
Recursive: if $x \in S$, then $3x \in S$

Fibonacci numbers
Basis: $(0, 0) \in S$, $(1, 1) \in S$
Examples of recursively defined sets

Natural numbers
  Basis: \( 0 \in S \)
  Recursive: if \( n \in S \), then \( n + 1 \in S \)

Even natural numbers
  Basis: \( 0 \in S \)
  Recursive: if \( x \in S \), then \( x + 2 \in S \)

Powers of 3
  Basis: \( 1 \in S \)
  Recursive: if \( x \in S \), then \( 3x \in S \)

Fibonacci numbers
  Basis: \((0, 0) \in S, (1, 1) \in S \)
  Recursive: if \((n - 1, x) \in S \) and \((n - 2, y) \in S \), then \((n, x + y) \in S \)
More examples of recursively defined sets

Strings

An alphabet $\Sigma$ is any finite set of characters.
The set $\Sigma^*$ of strings over the alphabet $\Sigma$ is defined as follows.

**Basis:** $\varepsilon \in \Sigma^*$, where $\varepsilon$ is the empty string.

**Recursive:** if $w \in \Sigma^*$ and $a \in \Sigma$, then $wa \in \Sigma^*$
More examples of recursively defined sets

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Palindromes (strings that are the same forwards and backwards)
More examples of recursively defined sets

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Palindromes (strings that are the same forwards and backwards)
Basis: $\varepsilon \in S$ and $a \in S$ for every $a \in \Sigma$
More examples of recursively defined sets

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The set $\Sigma^*$ of *strings* over the alphabet $\Sigma$ is defined as follows.

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**Recursive:** if $w \in \Sigma^*$ and $a \in \Sigma$, then $wa \in \Sigma^*$

**Palindromes** (strings that are the same forwards and backwards)

**Basis:** $\epsilon \in S$ and $a \in S$ for every $a \in \Sigma$

**Recursive:** if $p \in S$, then $apa \in S$ for every $a \in \Sigma$
More examples of recursively defined sets

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All binary strings with no 1’s before 0’s
More examples of recursively defined sets

Strings

An *alphabet* \( \Sigma \) is any finite set of characters.
The set \( \Sigma^* \) of *strings* over the alphabet \( \Sigma \) is defined as follows.
**Basis:** \( \varepsilon \in \Sigma^* \), where \( \varepsilon \) is the empty string.
**Recursive:** if \( w \in \Sigma^* \) and \( a \in \Sigma \), then \( wa \in \Sigma^* \)

Palindromes (strings that are the same forwards and backwards)

**Basis:** \( \varepsilon \in S \) and \( a \in S \) for every \( a \in \Sigma \)
**Recursive:** if \( p \in S \), then \( apa \in S \) for every \( a \in \Sigma \)

All binary strings with no 1’s before 0’s

**Basis:** \( \varepsilon \in S \)
More examples of recursively defined sets

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An alphabet $\Sigma$ is any finite set of characters.
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Basis: $\epsilon \in S$ and $a \in S$ for every $a \in \Sigma$
Recursive: if $p \in S$, then $apa \in S$ for every $a \in \Sigma$

All binary strings with no 1’s before 0’s
Basis: $\epsilon \in S$
Recursive: if $x \in S$, then $0x \in S$ and $x1 \in S$
Functions on recursively defined sets

Length

\[ \text{len}(\varepsilon) = 0 \]
\[ \text{len}(wa) = \text{len}(w) + 1 \text{ for } w \in \Sigma^*, a \in \Sigma \]

Define \( \Sigma^* \) by

- **Basis:** \( \varepsilon \in \Sigma^* \), where \( \varepsilon \) is the empty string.
- **Recursive:** if \( w \in \Sigma^* \) and \( a \in \Sigma \), then \( wa \in \Sigma^* \).
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\[\text{len}(\varepsilon) = 0\]
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Reversal

Define \(\Sigma^*\) by

Basis: \(\varepsilon \in \Sigma^*\), where \(\varepsilon\) is the empty string.
Recursive: if \(w \in \Sigma^*\) and \(a \in \Sigma\), then \(wa \in \Sigma^*\)
Functions on recursively defined sets

Length

\[
\begin{align*}
\text{len}(\varepsilon) &= 0 \\
\text{len}(wa) &= \text{len}(w) + 1 \text{ for } w \in \Sigma^*, a \in \Sigma
\end{align*}
\]

Reversal

\[
\varepsilon^R = \varepsilon
\]

Define \(\Sigma^*\) by

\begin{itemize}
  \item Basis: \(\varepsilon \in \Sigma^*\), where \(\varepsilon\) is the empty string.
  \item Recursive: if \(w \in \Sigma^*\) and \(a \in \Sigma\), then \(wa \in \Sigma^*\)
\end{itemize}
Functions on recursively defined sets

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\text{len}(\varepsilon) = 0
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\text{len}(wa) = \text{len}(w) + 1 \text{ for } w \in \Sigma^*, a \in \Sigma
\]

Reversal
\[
\varepsilon^R = \varepsilon
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\[
(wa)^R = aw^R \text{ for } w \in \Sigma^*, a \in \Sigma
\]

Define \(\Sigma^*\) by
Basis: \(\varepsilon \in \Sigma^*\), where \(\varepsilon\) is the empty string.
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Concatenation

Define \( \Sigma^* \) by

Basis: \( \varepsilon \in \Sigma^* \), where \( \varepsilon \) is the empty string.

Recursive: if \( w \in \Sigma^* \) and \( a \in \Sigma \), then \( wa \in \Sigma^* \)
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Reversal

\[ \varepsilon^R = \varepsilon \]
\[ (wa)^R = aw^R \text{ for } w \in \Sigma^*, a \in \Sigma \]

Concatenation

\[ x \cdot \varepsilon = x \text{ for } x \in \Sigma^* \]

Define \( \Sigma^* \) by

**Basis:** \( \varepsilon \in \Sigma^* \), where \( \varepsilon \) is the empty string.

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Functions on recursively defined sets

Length
\[
\text{len}(\varepsilon) = 0 \\
\text{len}(wa) = \text{len}(w) + 1 \quad \text{for} \quad w \in \Sigma^*, \ a \in \Sigma
\]

Reversal
\[
\varepsilon^R = \varepsilon \\
(wa)^R = aw^R \quad \text{for} \quad w \in \Sigma^*, \ a \in \Sigma
\]

Concatenation
\[
x \cdot \varepsilon = x \quad \text{for} \quad x \in \Sigma^* \\
x \cdot (wa) = (x \cdot w)a \quad \text{for} \quad x, w \in \Sigma^*, \ a \in \Sigma
\]

Define \(\Sigma^*\) by

Basis: \(\varepsilon \in \Sigma^*\), where \(\varepsilon\) is the empty string.

Recursive: if \(w \in \Sigma^*\) and \(a \in \Sigma\), then \(wa \in \Sigma^*\)
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Number of c’s in a string

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Number of c’s in a string
\[ \#_c(\varepsilon) = 0 \]

Define \( \Sigma^* \) by
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Reversal
$$\varepsilon^R = \varepsilon$$
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Concatenation
$$x \cdot \varepsilon = x \text{ for } x \in \Sigma^*$$
$$x \cdot (wa) = (x \cdot w)a \text{ for } x, w \in \Sigma^*, a \in \Sigma$$

Number of c’s in a string
$$\#_c(\varepsilon) = 0$$
$$\#_c(wa) = \#_c(w) + 1 \text{ for } w \in \Sigma^*$$

Define $\Sigma^*$ by
Basis: $\varepsilon \in \Sigma^*$, where $\varepsilon$ is the empty string.
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Concatenation
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x \cdot \varepsilon = x \text{ for } x \in \Sigma^* \\
x \cdot (wa) = (x \cdot w)a \text{ for } x, w \in \Sigma^*, a \in \Sigma
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Number of c's in a string
\[
#_c(\varepsilon) = 0 \\
#_c(wc) = #_c(w) + 1 \text{ for } w \in \Sigma^* \\
#_c(wa) = #_c(w) \text{ for } w \in \Sigma^*, a \in \Sigma, a \neq c
\]

Define $\Sigma^*$ by
Basis: $\varepsilon \in \Sigma^*$, where $\varepsilon$ is the empty string.
Recursive: if $w \in \Sigma^*$ and $a \in \Sigma$, then $wa \in \Sigma^*$
Rooted binary trees and functions on them

Rooted binary trees

Basis: \( \bullet \in S \)

Recursive: if \( L \in S \) and \( R \in S \), then \( \text{Tree}(\bullet, L, R) \in S \)
Rooted binary trees and functions on them

Rooted binary trees
  Basis: \( \bullet \in S \)
  Recursive: if \( L \in S \) and \( R \in S \), then \( \text{Tree}(\bullet, L, R) \in S \)

Size of a rooted binary tree

\[ L \quad \bullet \quad R \]

\[ \bullet \in S \]
Rooted binary trees and functions on them

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\[ | \bullet | = 1 \]
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Size of a rooted binary tree
  \( |\bullet| = 1 \)
  \( |\text{Tree}(\bullet, L, R)| = 1 + |L| + |R| \)
Rooted binary trees and functions on them

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   Basis: $\bullet \in S$
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Size of a rooted binary tree

$| \bullet | = 1$

$| \text{Tree}(\bullet, L, R)| = 1 + |L| + |R|$

Height of a rooted binary tree
Rooted binary trees and functions on them

Rooted binary trees

Basis: $\bullet \in S$

Recursive: if $L \in S$ and $R \in S$, then $\text{Tree}(\bullet, L, R) \in S$

Size of a rooted binary tree

$|\bullet| = 1$

$|	ext{Tree}(\bullet, L, R)| = 1 + |L| + |R|$

Height of a rooted binary tree

$[\bullet] = 0$
Rooted binary trees and functions on them

Rooted binary trees
  Basis: $\bullet \in S$
  Recursive: if $L \in S$ and $R \in S$, then $\text{Tree}(\bullet, L, R) \in S$

Size of a rooted binary tree
  $|\bullet| = 1$
  $|\text{Tree}(\bullet, L, R)| = 1 + |L| + |R|$

Height of a rooted binary tree
  $[\bullet] = 0$
  $[\text{Tree}(\bullet, L, R)] = 1 + \max([L], [R])$
Structural induction
A method for proving properties of recursive structures.
How can we prove properties of recursive structures?

Suppose that $S$ is a recursively defined set.

And we want to prove that every element of $S$ satisfies a predicate $P$.

Can we use ordinary induction to prove $\forall x \in S. P(x)$?
How can we prove properties of recursive structures?

Suppose that \( S \) is a recursively defined set.
And we want to prove that every element of \( S \) satisfies a predicate \( P \).

Can we use ordinary induction to prove \( \forall x \in S . P(x) \)?
Yes! Define \( Q(n) \) to be “for all \( x \in S \) that can be constructed in at most \( n \) recursive steps, \( P(x) \) is true.”
How can we prove properties of recursive structures?

Suppose that $S$ is a recursively defined set. And we want to prove that every element of $S$ satisfies a predicate $P$.

Can we use ordinary induction to prove $\forall x \in S. \ P(x)$?

Yes! Define $Q(n)$ to be “for all $x \in S$ that can be constructed in at most $n$ recursive steps, $P(x)$ is true.”

But this proof would be long and cumbersome to do! So we use structural induction instead.

- Follows from ordinary induction (on $Q$), while providing a more convenient proof template for reasoning about recursive structures.
- As powerful as ordinary induction, which is just structural induction applied to the recursively defined set of natural numbers.
Proving $\forall x \in S. \, P(x)$ by structural induction

1. Let $P(x)$ be [definition of $P(x)$].
   We will show that $P(x)$ is true for every $x \in S$ by structural induction.

2. Base cases:
   [Proof of $P(s_0), \ldots, P(s_m)$.]

3. Inductive hypothesis:
   Assume that $P(y_0), \ldots, P(y_k)$ are true for some arbitrary $y_0, \ldots, y_k \in S$.

4. Inductive step:
   We want to prove that $P(y)$ is true.
   [Proof of $P(y)$. The proof must invoke the structural inductive hypothesis.]

5. The result follows for all $x \in S$ by structural induction.
Proving $\forall x \in S. P(x)$ by structural induction

1. Let $P(x)$ be \textit{definition of $P(x)$}.
   We will show that $P(x)$ is true for every $x \in S$ by structural induction.

2. Base cases:
   \textit{Proof of $P(s_0), \ldots, P(s_m)$.}

3. Inductive hypothesis:
   Assume that $P(y_0), \ldots, P(y_k)$ are true for some arbitrary $y_0, \ldots, y_k \in S$.

4. Inductive step:
   We want to prove that $P(y)$ is true.
   \textit{Proof of $P(y)$. The proof must invoke the structural inductive hypothesis.}

5. The result follows for all $x \in S$ by structural induction.

Recursive definition of $S$

Basis step:
$s_0 \in S, \ldots, s_m \in S$.

Recursive step:
if $y_0, \ldots, y_k \in S$, then $y \in S$.

If the recursive step of $S$ includes multiple rules for constructing new elements from existing elements, then

3. \textbf{assume $P$} for the existing elements in every rule, and

4. \textbf{prove $P$} for the new element in every rule.
Using structural induction

Example proofs about recursively defined numbers, strings, and trees.
Prove that every \( x \in S \) is divisible by 3

Define \( S \) by

**Basis:** \( 6 \in S, \ 15 \in S \).

**Recursive:** if \( x, y \in S \), then \( x + y \in S \).
Prove that every $x \in S$ is divisible by 3

Let $P(x)$ be $3|x$.
We will show that $P(x)$ is true for every $x \in S$ by structural induction.

Define $S$ by
- **Basis:** $6 \in S, 15 \in S$.
- **Recursive:** if $x, y \in S$, then $x + y \in S$. 

① Let $P(x)$ be $3|x$.

We will show that $P(x)$ is true for every $x \in S$ by structural induction.

② Define $S$ by

- **Basis:** $6 \in S, 15 \in S$.
- **Recursive:** if $x, y \in S$, then $x + y \in S$. 

Prove that every $x \in S$ is divisible by 3

1. Let $P(x)$ be $3|x$.
   We will show that $P(x)$ is true for every $x \in S$ by structural induction.
2. Base cases ($x = 6$, $x = 15$):
   $3|6$ so $P(6)$ holds, and $3|15$ so $P(15)$ holds.

Define $S$ by
   Basis: $6 \in S$, $15 \in S$.
   Recursive: if $x, y \in S$, then $x + y \in S$. 
Prove that every \( x \in S \) is divisible by 3

1. Let \( P(x) \) be \( 3|x \).
   We will show that \( P(x) \) is true for every \( x \in S \) by structural induction.

2. Base cases (\( x = 6, x = 15 \)):
   \( 3|6 \) so \( P(6) \) holds, and \( 3|15 \) so \( P(15) \) holds.

3. Inductive hypothesis:
   Assume that \( P(x), P(y) \) are true for some arbitrary \( x, y \in S \).

Define \( S \) by
   
   Basis: \( 6 \in S, 15 \in S. \)
   
   Recursive: if \( x, y \in S \), then \( x + y \in S. \)
Prove that every $x \in S$ is divisible by 3

1. Let $P(x)$ be $3|x$.
   We will show that $P(x)$ is true for every $x \in S$ by structural induction.

2. **Base cases ($x = 6, x = 15$):**
   - $3|6$ so $P(6)$ holds, and $3|15$ so $P(15)$ holds.

3. **Inductive hypothesis:**
   Assume that $P(x), P(y)$ are true for some arbitrary $x, y \in S$.

4. **Inductive step:**
   We want to prove that $P(x + y)$ is true.

**Define $S$ by**

- **Basis:** $6 \in S, 15 \in S$.
- **Recursive:** if $x, y \in S$, then $x + y \in S$. 
Prove that every $x \in S$ is divisible by 3

① Let $P(x)$ be $3|x$.
    We will show that $P(x)$ is true for every $x \in S$ by structural induction.

② Base cases ($x = 6, x = 15$):
    3|6 so $P(6)$ holds, and 3|15 so $P(15)$ holds.

③ Inductive hypothesis:
    Assume that $P(x), P(y)$ are true for some arbitrary $x, y \in S$.

④ Inductive step:
    We want to prove that $P(x + y)$ is true.
    By the inductive hypothesis, 3|x and 3|y, so $x = 3i$ and $y = 3j$ for some $i, j \in \mathbb{Z}$.

Define $S$ by
    Basis: $6 \in S, 15 \in S$.
    Recursive: if $x, y \in S$, then $x + y \in S$. 
Prove that every $x \in S$ is divisible by 3

① Let $P(x)$ be $3|x$.
We will show that $P(x)$ is true for every $x \in S$ by structural induction.

② Base cases ($x = 6, x = 15$):
3|6 so $P(6)$ holds, and 3|15 so $P(15)$ holds.

③ Inductive hypothesis:
Assume that $P(x), P(y)$ are true for some arbitrary $x, y \in S$.

④ Inductive step:
We want to prove that $P(x + y)$ is true.
By the inductive hypothesis, $3|x$ and $3|y$, so $x = 3i$ and $y = 3j$ for some $i, j \in \mathbb{Z}$. Therefore,
$x + y = 3i + 3j = 3(i + j)$ so $3|(x + y)$.

Define $S$ by
Basis: $6 \in S, 15 \in S$.
Recursive: if $x, y \in S$, then $x + y \in S$. 
Prove that every \( x \in S \) is divisible by 3

1. Let \( P(x) \) be \( 3|x \).
   
   We will show that \( P(x) \) is true for every \( x \in S \) by structural induction.

2. Base cases (\( x = 6, x = 15 \)):
   
   \( 3|6 \) so \( P(6) \) holds, and \( 3|15 \) so \( P(15) \) holds.

3. Inductive hypothesis:
   
   Assume that \( P(x), P(y) \) are true for some arbitrary \( x, y \in S \).

4. Inductive step:
   
   We want to prove that \( P(x + y) \) is true.
   
   By the inductive hypothesis, \( 3|x \) and \( 3|y \), so \( x = 3i \) and \( y = 3j \) for some \( i, j \in \mathbb{Z} \). Therefore,
   
   \( x + y = 3i + 3j = 3(i + j) \) so \( 3|(x + y) \). Hence, \( P(x + y) \) is true.

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- Basis: \( 6 \in S, 15 \in S \).
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Prove that every $x \in S$ is divisible by 3

1. Let $P(x)$ be $3|x$.
   We will show that $P(x)$ is true for every $x \in S$ by structural induction.

2. Base cases ($x = 6, x = 15$):
   $3|6$ so $P(6)$ holds, and $3|15$ so $P(15)$ holds.

3. Inductive hypothesis:
   Assume that $P(x), P(y)$ are true for some arbitrary $x, y \in S$.

4. Inductive step:
   We want to prove that $P(x + y)$ is true.
   By the inductive hypothesis, $3|x$ and $3|y$, so $x = 3i$ and $y = 3j$ for some $i, j \in \mathbb{Z}$. Therefore,
   $x + y = 3i + 3j = 3(i + j)$ so $3|(x + y)$. Hence, $P(x + y)$ is true.

5. The result follows for all $x \in S$ by structural induction.

Define $S$ by
Basis: $6 \in S, 15 \in S$.
Recursive: if $x, y \in S$, then $x + y \in S$. 
Prove \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

Define \( \Sigma^* \) by

Basis: \( \epsilon \in \Sigma^* \).

Recursive:
if \( w \in \Sigma^* \) and \( a \in \Sigma \),
then \( wa \in \Sigma^* \)

Length
\( \text{len}(\epsilon) = 0 \)
\( \text{len}(wa) = \text{len}(w) + 1 \)

Concatenation
\( x \cdot \epsilon = x \)
\( x \cdot (wa) = (x \cdot w)a \)
Prove \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

What object \((x\ or \ y)\) to do structural induction on?

Define \( \Sigma^* \) by

- **Basis:** \( \epsilon \in \Sigma^* \).
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Prove \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

1. Let \( P(y) \) be \( \forall x \in \Sigma^*. \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \).
   We will show that \( P(y) \) is true for every \( y \in \Sigma^* \) by structural induction.

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1. Let $P(y)$ be $\forall x \in \Sigma^*. \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$.
   We will show that $P(y)$ is true for every $y \in \Sigma^*$ by structural induction.

2. Base case ($y = \varepsilon$):
   For every $x \in \Sigma^*$, $\text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ since $\text{len}(\varepsilon) = 0$. So $P(\varepsilon)$ is true.

Define $\Sigma^*$ by
- Basis: $\varepsilon \in \Sigma^*$.
- Recursive:
  if $w \in \Sigma^*$ and $a \in \Sigma$, then $wa \in \Sigma^*$

Length
- $\text{len}(\varepsilon) = 0$
- $\text{len}(wa) = \text{len}(w) + 1$

Concatenation
- $x \cdot \varepsilon = x$
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   We will show that \( P(y) \) is true for every \( y \in \Sigma^* \) by structural induction.

2. **Base case** \( (y = \varepsilon) \):
   For every \( x \in \Sigma^* \), \( \text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon) \) since \( \text{len}(\varepsilon) = 0 \). So \( P(\varepsilon) \) is true.

3. **Inductive hypothesis:**
   Assume that \( P(w) \) is true for some arbitrary \( w \in \Sigma^* \).

---

**Define \( \Sigma^* \) by**

- **Basis:** \( \varepsilon \in \Sigma^* \).
- **Recursive:** if \( w \in \Sigma^* \) and \( a \in \Sigma \), then \( wa \in \Sigma^* \).

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3. Inductive hypothesis:
   Assume that \( P(w) \) is true for some arbitrary \( w \in \Sigma^* \).

4. Inductive step:
   We want to prove that \( P(wa) \) is true for every \( a \in \Sigma \).

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- Basis: \( \varepsilon \in \Sigma^* \).
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  if \( w \in \Sigma^* \) and \( a \in \Sigma \), then \( wa \in \Sigma^* \)
  - Length
    - \( \text{len}(\varepsilon) = 0 \)
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    - \( x \cdot \varepsilon = x \)
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1. Let $P(y)$ be $\forall x \in \Sigma^*. \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$.
   We will show that $P(y)$ is true for every $y \in \Sigma^*$ by structural induction.

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   For every $x \in \Sigma^*$, $\text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ since $\text{len}(\varepsilon) = 0$. So $P(\varepsilon)$ is true.

3. Inductive hypothesis:
   Assume that $P(w)$ is true for some arbitrary $w \in \Sigma^*$.

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   We want to prove that $P(wa)$ is true for every $a \in \Sigma$.
   Let $a \in \Sigma$ and $x \in \Sigma^*$ be arbitrary. Then

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   For every $x \in \Sigma^*$, $\text{len}(x \cdot \epsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\epsilon)$ since $\text{len}(\epsilon) = 0$. So $P(\epsilon)$ is true.

3. Inductive hypothesis:
   Assume that $P(w)$ is true for some arbitrary $w \in \Sigma^*$.

4. Inductive step:
   We want to prove that $P(wa)$ is true for every $a \in \Sigma$.
   Let $a \in \Sigma$ and $x \in \Sigma^*$ be arbitrary. Then
   $$\text{len}(x \cdot wa) = \text{len}((x \cdot w)a)$$
   by defn of $\cdot$
   $$= \text{len}(x \cdot w) + 1$$
   by defn of $\text{len}$
   $$= \text{len}(x) + \text{len}(w) + 1$$
   by IH
   $$= \text{len}(x) + \text{len}(wa)$$
   by defn of $\text{len}$

Define $\Sigma^*$ by
Basis: $\epsilon \in \Sigma^*$.
Recursive:
if $w \in \Sigma^*$ and $a \in \Sigma$, then $wa \in \Sigma^*$

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Prove \( \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \) for all \( x, y \in \Sigma^* \)

1. Let \( P(y) \) be \( \forall x \in \Sigma^*. \text{len}(x \cdot y) = \text{len}(x) + \text{len}(y) \). We will show that \( P(y) \) is true for every \( y \in \Sigma^* \) by structural induction.

2. Base case (\( y = \varepsilon \)):
   For every \( x \in \Sigma^* \), \( \text{len}(x \cdot \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon) \) since \( \text{len}(\varepsilon) = 0 \). So \( P(\varepsilon) \) is true.

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   Assume that \( P(w) \) is true for some arbitrary \( w \in \Sigma^* \).

4. Inductive step:
   We want to prove that \( P(wa) \) is true for every \( a \in \Sigma \).
   Let \( a \in \Sigma \) and \( x \in \Sigma^* \) be arbitrary. Then
   \[
   \text{len}(x \cdot wa) = \text{len}((x \cdot wa)) \\
   = \text{len}(x \cdot w) + 1 \\n   = \text{len}(x) + \text{len}(w) + 1 \\n   = \text{len}(x) + \text{len}(wa) \\n   \]
   So \( \text{len}(x \cdot wa) = \text{len}(x) + \text{len}(wa) \) for all \( x \in \Sigma^* \), and \( P(wa) \) is true.

Define \( \Sigma^* \) by
Basis: \( \varepsilon \in \Sigma^* \).
Recursive:
if \( w \in \Sigma^* \) and \( a \in \Sigma \), then \( wa \in \Sigma^* \)

Length
\[
\text{len}(\varepsilon) = 0 \\
\text{len}(wa) = \text{len}(w) + 1
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Concatenation
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x \cdot \varepsilon = x \\
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   We want to prove that \( P(wa) \) is true for every \( a \in \Sigma \).
   Let \( a \in \Sigma \) and \( x \in \Sigma^* \) be arbitrary. Then
   \[
   \text{len}(x \cdot wa) = \text{len}((x \cdot wa)) \quad \text{by defn of } \cdot \\
   = \text{len}(x \cdot w) + 1 \quad \text{by defn of } \text{len} \\
   = \text{len}(x) + \text{len}(w) + 1 \quad \text{by IH} \\
   = \text{len}(x) + \text{len}(wa) \quad \text{by defn of } \text{len} \\
   \]
   So \( \text{len}(x \cdot wa) = \text{len}(x) + \text{len}(wa) \) for all \( x \in \Sigma^* \), and \( P(wa) \) is true.

5. The result follows for all \( y \in \Sigma^* \) by structural induction.
Prove $|t| \leq 2^{[t]+1} - 1$ for every rooted binary tree $t$

Define $S$ by

Basis: $\bullet \in S$.

Recursive:
if $L, R \in S$, then
Tree($\bullet, L, R$) $\in S$

Size
$|\bullet| = 1$
$|\text{Tree}(\bullet, L, R)| = 1 + |L| + |R|$

Height
$[\bullet] = 0$
$[\text{Tree}(\bullet, L, R)] = 1 + \max([L], [R])$
Prove $|t| \leq 2^{[t]+1} - 1$ for every rooted binary tree $t$

Let $P(t)$ be $|t| \leq 2^{[t]+1} - 1$.

We will show that $P(t)$ is true for every $t \in S$ by structural induction.

Define $S$ by

**Basis:** $\bullet \in S$.

**Recursive:**

if $L, R \in S$, then

$\text{Tree}(\bullet, L, R) \in S$

**Size**

$|\bullet| = 1$

$|\text{Tree}(\bullet, L, R)| = 1 + |L| + |R|$

**Height**

$[\bullet] = 0$

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Prove $|t| \leq 2^{[t]+1} - 1$ for every rooted binary tree $t$

1. Let $P(t)$ be $|t| \leq 2^{[t]+1} - 1$.
   We will show that $P(t)$ is true for every $t \in S$ by structural induction.

2. Base case ($t = \bullet$):
   
   $|\bullet| = 1 = 2^1 - 1 = 2^{0+1} - 1 = 2^{[\bullet]+1} - 1$ so $P(\bullet)$ is true.

Define $S$ by
- **Basis:** $\bullet \in S$.
- **Recursive:**
  - if $L, R \in S$, then $\text{Tree}(\bullet, L, R) \in S$

**Size**
- $|\bullet| = 1$
- $|\text{Tree}(\bullet, L, R)| = 1 + |L| + |R|$  

**Height**
- $[\bullet] = 0$
- $[\text{Tree}(\bullet, L, R)] = 1 + \max([L], [R])$
Prove $|t| \leq 2^{[t]+1} - 1$ for every rooted binary tree $t$

1. Let $P(t)$ be $|t| \leq 2^{[t]+1} - 1$.
   We will show that $P(t)$ is true for every $t \in S$ by structural induction.

2. Base case ($t = \cdot$):
   $|\cdot| = 1 = 2^1 - 1 = 2^{0+1} - 1 = 2^{[\cdot]+1} - 1$ so $P(\cdot)$ is true.

3. Inductive hypothesis:
   Assume that $P(L)$ and $P(R)$ are true for some arbitrary $L, R \in S$.

Define $S$ by

Basis: $\cdot \in S$.

Recursive:
if $L, R \in S$, then $\text{Tree}(\cdot, L, R) \in S$

Size
$|\cdot| = 1$
$|\text{Tree}(\cdot, L, R)| = 1 + |L| + |R|$

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$[\cdot] = 0$
$[\text{Tree}(\cdot, L, R)] = 1 + \max([L], [R])$
Prove $|t| \leq 2^{[t]+1} - 1$ for every rooted binary tree $t$

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   We will show that $P(t)$ is true for every $t \in S$ by structural induction.

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3. Inductive hypothesis:
   
   Assume that $P(L)$ and $P(R)$ are true for some arbitrary $L, R \in S$.

4. Inductive step:
   
   We want to prove that $P(\text{Tree}(\bullet, L, R))$ is true.

Define $S$ by

Basis: $\bullet \in S$.

Recursive:

if $L, R \in S$, then $\text{Tree}(\bullet, L, R) \in S$

Size

$|\bullet| = 1$

$|\text{Tree}(\bullet, L, R)| = 1 + |L| + |R|$

Height

$[\bullet] = 0$

$[\text{Tree}(\bullet, L, R)] = 1 + \max([L], [R])$
Prove \(|t| \leq 2^{[t]+1} - 1\) for every rooted binary tree \(t\)

1. Let \(P(t) = |t| \leq 2^{[t]+1} - 1\).
   We will show that \(P(t)\) is true for every \(t \in S\) by structural induction.

2. Base case (\(t = \bullet\)):
   \(|\bullet| = 1 = 2^1 - 1 = 2^{0+1} - 1 = 2^{[\bullet]+1} - 1\) so \(P(\bullet)\) is true.

3. Inductive hypothesis:
   Assume that \(P(L)\) and \(P(R)\) are true for some arbitrary \(L, R \in S\).

4. Inductive step:
   We want to prove that \(P(\text{Tree}(\bullet, L, R))\) is true.
   \begin{align*}
   |\text{Tree}(\bullet, L, R)| &= 1 + |L| + |R| \\
   &\leq 1 + (2^{[L]+1} - 1) + (2^{[R]+1} - 1) \quad \text{by defn of } || \\
   &\leq 2^{[L]+1} + 2^{[R]+1} - 1 \quad \text{by IH} \\
   &\leq 2(2^{\max([L],[R])+1}) - 1 \quad \text{algebra} \\
   &= 2(2^{\text{Tree}(\bullet,L,R)}) - 1 \quad \text{by defn of max} \\
   &= 2^{\text{Tree}(\bullet,L,R)+1} - 1 \quad \text{by defn of } [|] \\
   &\leq 2^{[t]+1} - 1 \quad \text{which is the desired result.}
   \end{align*}
Prove $|t| \leq 2^{[t]+1} - 1$ for every rooted binary tree $t$

1. Let $P(t)$ be $|t| \leq 2^{[t]+1} - 1$.
   
   We will show that $P(t)$ is true for every $t \in S$ by structural induction.

2. Base case ($t = \bullet$):
   
   $|\bullet| = 1 = 2^1 - 1 = 2^{0+1} - 1 = 2^{[\bullet]+1} - 1$ so $P(\bullet)$ is true.

3. Inductive hypothesis:
   
   Assume that $P(L)$ and $P(R)$ are true for some arbitrary $L, R \in S$.

4. Inductive step:
   
   We want to prove that $P(\text{Tree}(\bullet, L, R))$ is true.
   
   $|\text{Tree}(\bullet, L, R)| = 1 + |L| + |R|$ by defn of $||$
   
   $\leq 1 + (2^{[L]+1} - 1) + (2^{[R]+1} - 1)$ by IH
   
   $\leq 2^{[L]+1} + 2^{[R]+1} - 1$ algebra
   
   $\leq 2(2^{\max([L],[R])+1}) - 1$ by defn of max
   
   $= 2(2^{[\text{Tree}(\bullet, L, R)]}) - 1$ by defn of $[]$
   
   $= 2^{[\text{Tree}(\bullet, L, R)]+1} - 1$ which is the desired result.

5. The result follows for all $t \in S$ by structural induction.
Summary

To define a set recursively, specify its basis and recursive step.
   Recursive set definitions assume the exclusion rule.
   We use recursive functions to operate on elements of recursive sets.

Use structural induction to prove properties of recursive structures.
   Structural induction follows from ordinary induction but is easier to use.

To prove \( \forall x \in S. P(x) \) using structural induction:
   Show that \( P \) holds for the elements in the basis step of \( S \).
   Assume \( P \) for every existing element of \( S \) named in the recursive step.
   Prove \( P \) for every new element of \( S \) created in the recursive step.