1. Been Square, Done That (16 points)

Let the domain of discourse be the real numbers ($\mathbb{R}$). We define the predicate $\text{OnLine}(a, b, x, y)$ to be true iff $(x, y)$ lies on the line with slope $a$ and intercept $b$ (i.e., iff $ax + b = y$) and the predicate $\text{InSquare}(u, v, r, x, y)$ to be true iff $r > 0$ and $(x, y)$ lies inside of an $r \times r$ square, which has its bottom left corner at $(u, v)$.

Give an English proof of the following claim:

$$\forall u. \forall v. \forall r. (r > 0) \rightarrow \exists a. \exists b. \exists x. \exists y. x \neq u \land y \neq v \land \text{InSquare}(u, v, r, x, y) \land \text{OnLine}(a, b_1, u, v) \land \text{OnLine}(a, b_1, u + r, v + r) \land \text{OnLine}(-a, b_2, u, v + r) \land \text{OnLine}(-a, b_2, u + r, v) \land \text{OnLine}(a, b_1, x, y) \land \text{OnLine}(-a, b_2, x, y)$$

2. Intersect Repellent (20 points)

Prove, via an English proof or using translations of Boolean algebra equivalences to set theory, each of the following claims for arbitrary sets $A$, $B$, and $C$.

(a) [10 Points] $(A \cup B) \cap (B \setminus A) \cap (B \setminus C) = A \cap B \cap C$

(b) [10 Points] $(B \setminus A) \cap (C \setminus A) = (B \cap C) \setminus A$

3. All Set (16 points)

Prove or disprove the following statements. If a proof is given, it must be directly from the set theory definitions (not translations of Boolean algebra equivalences) and in English (not formal).

(a) [8 Points] For any two sets $R$ and $S$, it holds that $\mathcal{P}(R) \cup \mathcal{P}(S) \subseteq \mathcal{P}(R \cup S)$.

(b) [8 Points] For any two sets $R$ and $S$, it holds that $\mathcal{P}(R \cup S) \subseteq \mathcal{P}(R) \cup \mathcal{P}(S)$.

4. Keeping Up With the Cartesians (16 points)

Let $A$, $B$, and $C$ be non-empty sets. Give an English proof that $(A \times B = A \times C) \rightarrow (B = C)$.

What happens if $A$ is empty?

5. A Mod and a Wink (16 points)

Let $a, b$ be integers and $c, m$ be positive integers. Give an English proof that $a \equiv b \pmod{m}$ if and only if $ca \equiv cb \pmod{cm}$. 
6. A Whale of a Good Prime (16 points)
Give an English proof that, for any prime $p > 2$, we have $p^2 + 1 \equiv 2 \pmod{4}$.

7. Extra Credit: Matchmaking (0 points)
In this problem, you will show that given $n$ red points and $n$ blue points in the plane such that no three points lie on a common line, it is possible to draw line segments between red-blue pairs so that all the pairs are matched and none of the line segments intersect. Assume that there are $n$ red and $n$ blue points fixed in the plane.

A matching $M$ is a collection of $n$ line segments connecting distinct red-blue pairs. The total length of a matching $M$ is the sum of the lengths of the line segments in $M$. Say that a matching $M$ is minimal if there is no matching with a smaller total length.

Let $\text{IsMinimal}(M)$ be the predicate that is true precisely when $M$ is a minimal matching. Let $\text{HasCrossing}(M)$ be the predicate that is true precisely when there are two line segments in $M$ that cross each other.

Give an argument in English explaining why there must be at least one matching $M$ so that $\text{IsMinimal}(M)$ is true, i.e.

$$\exists M (\text{IsMinimal}(M))$$

Give an argument in English explaining why

$$\forall M (\text{HasCrossing}(M) \rightarrow \neg \text{IsMinimal}(M))$$

Now use the two results above to give a proof of the statement:

$$\exists M \neg \text{HasCrossing}(M).$$