CSE 311: Foundations of Computing I

Induction Solutions

Induction

(a) Prove that $9 \mid n^3 + (n+1)^3 + (n+2)^3$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

Induction Hypothesis: Assume that $9 \mid j^3 + (j+1)^3 + (j+2)^3$ for some arbitrary integer $j > 1$. Note that this is equivalent to assuming that $j^3 + (j+1)^3 + (j+2)^3 = 9k$ for some integer k.

Induction Step: Goal: Show $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$

Now

$$\begin{aligned}
(j+1)^3 + (j+2)^3 + (j+3)^3 &= (j+3)^3 + 9k - j^3 \text{ for some integer } k \quad \text{[Induction Hypothesis]} \\
&= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\
&= 9j^2 + 27j + 27 + 9k \\
&= 9(j^2 + 3j + 3 + k)
\end{aligned}$$

So $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$, so $P(j) \to P(j+1)$ for an arbitrary integer $j > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

(b) Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Solution:

Let $P(n)$ be "$6n + 6 < 2^n$". We will prove $P(n)$ for all integers $n \geq 6$ by induction.

Base Case $(n = 6)$**:** $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

Induction Hypothesis: Assume that $6j + 6 < 2^j$ for some arbitrary integer $j \geq 6$.

Induction Step: Goal: Show $6(j+1) + 6 < 2^{j+1}$

Now

$$\begin{aligned}
6(j+1) + 6 &= 6j + 6 + 6 \\
&< 2^j + 6 && \text{[Induction Hypothesis]} \\
&< 2^j + 2^j && \text{[Since } 2^j > 6 \text{, since } j \geq 6\text{]} \\
&< 2 \cdot 2^j \\
&< 2^{j+1}
\end{aligned}$$

So $P(j) \to P(j+1)$ for an arbitrary integer $j \geq 6$.

Conclusion: $P(n)$ holds for all integers $n \geq 6$ by induction.

(c) Define

$$H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for $n \in \mathbb{N}$.

Solution:

We define H_i more formally as $\sum_{k=1}^{i} \frac{1}{k}$. Let $P(n)$ be "$H_{2^n} \geq 1 + \frac{n}{2}$". We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

Base Case $(n = 0)$**:** $H_{2^0} = H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq 1 + \frac{0}{2}$, so $P(0)$ holds.

Induction Hypothesis: Assume that $H_{2^j} \geq 1 + \frac{j}{2}$ for some arbitrary integer $j \in \mathbb{N}$.

Induction Step: Goal: Show $H_{2^{j+1}} \geq 1 + \dfrac{j+1}{2}$

Now

$$
\begin{aligned}
H_{2^{j+1}} &= \sum_{k=1}^{2^{j+1}} \frac{1}{k} \\
&= \sum_{k=1}^{2^j} \frac{1}{k} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \\
&\geq 1 + \frac{j}{2} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} && \text{[Induction Hypothesis]} \\
&\geq 1 + \frac{j}{2} + 2^j \cdot \frac{1}{2^{j+1}} && \text{[There are } 2^j \text{ terms in } [2^j+1, 2^{j+1}] \text{ and each is at least } \frac{1}{2^{j+1}}] \\
&\geq 1 + \frac{j}{2} + \frac{2^j}{2^{j+1}} \\
&\geq 1 + \frac{j}{2} + \frac{1}{2} \\
&\geq 1 + \frac{j+1}{2}
\end{aligned}
$$

So $P(j) \to P(j+1)$ for an arbitrary integer $j \in \mathbb{N}$.

Conclusion: $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.

Strong Induction

(a) Prove that, for all $n \in \mathbb{N}$, every n has an unsigned binary representation.

Solution:

Let $P(n)$ be "n has an unsigned binary representation". We will prove $P(n)$ for all integers $n \in \mathbb{N}$ by induction.

Base Case $(n = 0)$**:** The unsigned binary representation of 0 is 0_2, so $P(0)$ holds.

Induction Hypothesis: Assume that $P(j)$ holds for all integers $0 \leq j \leq k$ for some arbitrary $k \in \mathbb{N}$.

Induction Step: Goal: Show $P(k+1)$ has an unsigned binary representation

Let 2^ℓ be the largest power of two not greater than $k+1$ (i.e. $\ell = \lfloor \log_2(n) \rfloor$). Let $r = k + 1 - 2^\ell$, the remainder.
Note that $r < 2^\ell < k$, so r has some binary representation r_2 [by the Induction Hypothesis].
Then $1r_2$ is the binary expansion for $k+1$.
So $P(0) \wedge P(1) \wedge \cdots \wedge P(k) \to P(k+1)$ for some arbitrary $k \in \mathbb{N}$.

Conclusion: $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.

(b) Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$f(0) = 0$$
$$f(1) = 1$$
$$f(n) = 2f(n-1) - f(n-2)$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n.

Solution:

Let $P(n)$ be "$f(n) = n$". We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on n.

Base Cases $(n = 0, n = 1)$**:** $f(0) = 0$ and $f(1) = 1$ by definition.

Induction Hypothesis: Assume that $P(0) \wedge P(1) \wedge \dots P(n-1)$ are true for some fixed but arbitrary $n - 1 \geq 1$.

Induction Step: We show $P(n)$:

$$\begin{aligned} f(n) &= 2f(n-1) - f(n-2) && \text{[Definition of } f\text{]} \\ &= 2(n-1) - (n-2) && \text{[Induction Hypothesis]} \\ &= n && \text{[Algebra]} \end{aligned}$$

Therefore, $P(n)$ is true for all $n \in \mathbb{N}$.