CSE 311: Foundations of Computing I

Section 5: Number Theory & Induction

0. More Number Theory

(a) Prove that if $n^2 + 1$ is a perfect square, where n is an integer, then n is even.

(b) Prove that if n is a positive integer such that the sum of the divisors of n is $n + 1$, then n is prime.

1. Induction

(a) Prove for all $n \in \mathbb{N}$ that if you have two groups of numbers, $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$, such that $\forall(i \in [n]).\ a_i \leq b_i$, then it must be that:

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i$$

(b) For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = \sum_{i=1}^{n} i^2.$$

For all $n \in \mathbb{N}$, prove that $S_n = \frac{1}{6}n(n+1)(2n+1)$.

(c) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. We showed in lecture that $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$\sum_{i=0}^{n} i^3 = \triangle_n^2$$