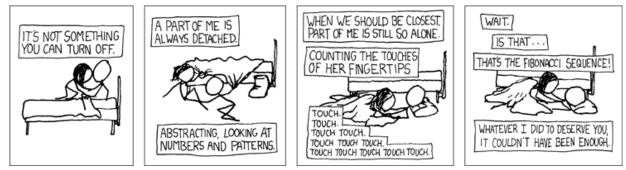
Adam Blank Spring 2017

CSE 311

Foundations of Computing I

CSE 311: Foundations of Computing

Lecture 17: Structural Induction



Strings

- An *alphabet* **Σ** is any finite set of characters
- The set **Σ*** is the set of *strings* over the alphabet **Σ**.

Adam

$$\Sigma^* = \varepsilon \mid \Sigma^* \sigma$$

A STRING is EMPTY or "STRING CHAR".

- **The set of strings is made up of:**
  - $\varepsilon \in \Sigma^*$ (ε is the empty string)
  - If $W \in \Sigma^*$, $\sigma \in \Sigma$, then $W\sigma \in \Sigma^*$

Palindromes

Palindromes are strings that are the same backwards and forwards (e.g. "abba", "tht", "neveroddoreven").

$$\text{Pal} = \varepsilon \mid \sigma \mid \sigma \,\text{Pal}\, \sigma$$

A PAL is EMPTY or CHAR or "CHAR PAL CHAR".

$abba = a(bb)a = a(b(\varepsilon)b)a$

Recursively Defined Programs (on Binary Strings)

$$\mathbf{B} = \varepsilon \mid 0 \mid 1 \mid B_0 + B_1$$

A BSTR is EMPTY, 0, 1, or "BSTR0 BSTR1".

Let's write a "reverse" function for binary strings.

$1+(0+(1+0))$

$$\text{rev} : B \to B$$

rev is a function that takes in a binary string and returns a binary string

$\text{rev}(\varepsilon) = \varepsilon$
$\text{rev}(0) = 0$
$\text{rev}(1) = 1$
$\text{rev}(a+b) = \boxed{\text{rev}(b) + \text{rev}(a)}$

Recursively Defined Programs (on Binary Strings)

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A BSTR is EMPTY, 0, 1, or "BSTR0 BSTR1".

Let's write a "reverse" function for binary strings.

$$
\begin{aligned}
&\text{rev} : B \to B \\
&\text{rev}(\varepsilon) = \varepsilon \\
&\text{rev}(0) = 0 \\
&\text{rev}(1) = 1 \\
&\text{rev}(a+b) = \text{rev}(b) + \text{rev}(a)
\end{aligned}
$$

Recursively Defined Programs (on Binary Strings)

$$\mathbf{B} = \varepsilon \mid 0 \mid 1 \mid B_0 + B_1$$

rev : B → B

rev(ε) = ε
rev(0) = 0
rev(1) = 1
rev(a + b) = rev(b) + rev(a)

Claim: For all binary strings **X**, rev(rev(X)) = X

Case ε: rev(rev(ε)) = rev(ε) = ε — **Def of rev**
Case 0: rev(rev(0)) = rev(0) = 0 — **Def of rev**
Case 1: rev(rev(1)) = rev(1) = 1 — **Def of rev**

Recursively Defined Programs (on Binary Strings)

$$\mathbf{B} = \varepsilon \mid 0 \mid 1 \mid B_0 + B_1$$

rev : B → B

rev(ε) = ε
rev(0) = 0
rev(1) = 1
rev(a + b) = rev(b) + rev(a)

Claim: For all binary strings **X**, rev(rev(X)) = X

Suppose rev(rev(a)) = a and rev(rev(b)) = b for some strings a, b.

Case $a + b$:
rev(rev(a + b)) = rev(rev(b) + rev(a)) = rev(rev(a)) + rev(rev(b)) $\overset{\text{by IH}}{=}$ a + b

Recursively Defined Programs (on Binary Strings)

$$\mathbf{B} = \varepsilon \mid 0 \mid 1 \mid B_0 + B_1$$

rev(ε) = ε
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Case $a + b$:
rev(rev(a + b)) = rev(rev(b) + rev(a)) — **Def of rev**
= rev(rev(a)) + rev(rev(b)) — **Def of rev**
= a + b — **By IH!**

Recursively Defined Programs (on Binary Strings)

$$\mathbf{B} = \varepsilon \mid 0 \mid 1 \mid B_0 + B_1$$

rev : B → B
rev(ε) = ε
rev(0) = 0
rev(1) = 1
rev(a + b) = rev(b) + rev(a)

Claim: For all binary strings **X**, rev(rev(X)) = X

We go by structural induction on B. Suppose rev(rev(a)) = a and rev(rev(b)) = b for some strings a, b.

Case ε: rev(rev(ε)) = rev(ε) = ε — **Def of rev**
Case 0: rev(rev(0)) = rev(0) = 0 — **Def of rev**
Case 1: rev(rev(1)) = rev(1) = 1 — **Def of rev**
Case $a + b$:
rev(rev(a + b)) = rev(rev(b) + rev(a)) — **Def of rev**
= rev(rev(a)) + rev(rev(b)) — **Def of rev**
= a + b — **By IH!**

Since the claim is true for all the cases, it's true for all binary strings.

All Binary Strings with no 1's before 0's

00 ✓
01 ✓
10 N
1001 N

$A = \varepsilon \mid 0 + A_0 \mid A_1 + 1$

All Binary Strings with no 1's before 0's

$$\mathbf{A} = \varepsilon \mid 0 + A_0 \mid A_1 + 1$$

A BIN is EMPTY or "0 BIN" or "BIN 1".

len : A → Int	
len(ε)	= 0
len(0 + a)	= 1 + len(a)
len(a + 1)	= 1 + len(a)

#0: A → Int	
#0(ε)	= 0
#0(0 + a)	= 1 + #0(a)
#0(a + 1)	= #0(a)

no1: A → A	
no1 (ε)	= ε
no1(0 + a)	= 0 + no1(a)
no1(a + 1)	= no1(a)

All Binary Strings with no 1's before 0's

$\mathbf{A} = \varepsilon \mid 0 + A_0 \mid A_1 + 1$

len : A → Int	#0: A → Int	no1: A → A
len(ε) = 0	#0(ε) = 0	no1 (ε) = ε
len(0 + a) = 1 + len(a)	#0(0 + a) = 1 + #0(a)	no1(0 + a) = 0 + no1(a)
len(a + 1) = 1 + len(a)	#0(a + 1) = #0(a)	no1(a + 1) = no1(a)

Claim: Prove that for all $x \in A$, len(no1(x)) = #0(x)

Case A = ε:

len(no1(ε)) = len(ε) by def of no1

= 0 by def of len

= #0(ε) by def of #0

All Binary Strings with no 1's before 0's

$\mathbf{A} = \varepsilon \mid 0 + A_0 \mid A_1 + 1$

len : A → Int	#0: A → Int	no1: A → A
len(ε) = 0	#0(ε) = 0	no1 (ε) = ε
len(0 + a) = 1 + len(a)	#0(0 + a) = 1 + #0(a)	no1(0 + a) = 0 + no1(a)
len(a + 1) = 1 + len(a)	#0(a + 1) = #0(a)	no1(a + 1) = no1(a)

Claim: Prove that for all $x \in A$, len(no1(x)) = #0(x)

We go by structural induction on A. Let A ∈ A be arbitrary.

Case A = ε:

len(no1(ε)) = len(ε) [Def of no1]

= 0 [Def of len]

= #0(ε) [Def of #0]

All Binary Strings with no 1's before 0's

$\mathbf{A} = \varepsilon \mid 0 + A_0 \mid A_1 + 1$

len : A → Int	#0: A → Int	no1: A → A
len(ε) = 0	#0(ε) = 0	no1 (ε) = ε
len(0 + a) = 1 + len(a)	#0(0 + a) = 1 + #0(a)	no1(0 + a) = 0 + no1(a)
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Claim: Prove that for all $x \in A$, len(no1(x)) = #0(x)

We go by structural induction on A. Let A ∈ A be arbitrary.

Suppose len(no1(x)) = #0(x) is true for some $x \in A$.

Case A = 0 + x:

len(no1(0+x)) = len(0 + no1(x)) by def no1

= 1 + len(no1(x)) by def len

= 1 + #0(x) by IH

= #0(0+x)

All Binary Strings with no 1's before 0's

$\mathbf{A} = \varepsilon \mid 0 + A_0 \mid A_1 + 1$

len : A → Int	#0: A → Int	no1: A → A
len(ε) = 0	#0(ε) = 0	no1 (ε) = ε
len(0 + a) = 1 + len(a)	#0(0 + a) = 1 + #0(a)	no1(0 + a) = 0 + no1(a)
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We go by structural induction on A. Let A ∈ A be arbitrary.

Suppose len(no1(x)) = #0(x) is true for some $x \in A$.

Case A = 0 + x:

len(no1(0 + x)) = len(0 + no1(x)) [Def of no1]

= 1 + len(no1(x)) [Def of len]

= 1 + #0(x) [By IH]

= #0(0 + x) [Def of #0]

All Binary Strings with no 1's before 0's

$\mathbf{A} = \varepsilon \mid 0 + A_0 \mid A_1 + 1$

len : A → Int	#0: A → Int	no1: A → A
len(ε) = 0	#0(ε) = 0	no1 (ε) = ε
len(0 + a) = 1 + len(a)	#0(0 + a) = 1 + #0(a)	no1(0 + a) = 0 + no1(a)
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Claim: Prove that for all $x \in A$, len(no1(x)) = #0(x)

We go by structural induction on A. Let A ∈ A be arbitrary.

Suppose len(no1(x)) = #0(x) is true for some $x \in A$.

Case A = x + 1:

len(no1(x+1)) = len(no1(x)) by def no1

= #0(x) by IH

= #(x+1) by def of #0

All Binary Strings with no 1's before 0's

$\mathbf{A} = \varepsilon \mid 0 + A_0 \mid A_1 + 1$

len : A → Int	#0: A → Int	no1: A → A
len(ε) = 0	#0(ε) = 0	no1 (ε) = ε
len(0 + a) = 1 + len(a)	#0(0 + a) = 1 + #0(a)	no1(0 + a) = 0 + no1(a)
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Claim: Prove that for all $x \in A$, len(no1(x)) = #0(x)

We go by structural induction on A. Let A ∈ A be arbitrary.

Suppose len(no1(x)) = #0(x) is true for some $x \in A$.

Case A = x + 1:

len(no1(x + 1)) = len(no1(x)) [Def of no1]

= #0(x) [By IH]

= #0(x + 1) [Def of #0]

Recursively Defined Programs (on Lists)

List = [ ] | a :: L

We'll assume a is an integer.

Write a function
len : List → Int
that computes the length of a list.

len([]) = 0
len(x :: L) = 1 + len(L)

Finish the function
append : (List, Int) → List
append([], i) = ... i::[]
append(a :: L, i) = ... a :: append(L, i)
which returns a list with i appended to the end

Recursively Defined Programs (on Lists)

List = [ ] | a :: L

We'll assume a is an integer.

len : List → Int
len([]) = 0
len(a :: L) = 1 + len(L)

append : (List, Int) → List
append([], i) = i::[]
append(a :: L, i) = a :: append(L, i)

Claim: For all lists **L**, and integers **i**, if len(L) = n, then len(append(L, i)) = n + 1.

Recursively Defined Programs (on Lists)

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len : List → Int
len([]) = 0
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Claim: For all lists **L**, and integers **i**, if len(L) = n, then len(append(L, i)) = n + 1.

Recursively Defined Programs (on Lists)

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append : (List, Int) → List
append([], i) = i::[]
append(a :: L, i) = a :: append(L, i)

Claim: For all lists **L**, and integers **i**, if len(L) = n, then len(append(L, i)) = n + 1.

We go by structural induction on List. Let i be an integer, and let L be a list. Suppose len(L) = n.

Case L = []:

len(append([], i))	= len(i::[])	[Def of append]
	= 1 + len([])	[Def of len]
	= 1 + 0	[Def of len]
	= 1	[Arithmetic]

Recursively Defined Programs (on Lists)

len : List → Int
len([]) = 0
len(a :: L) = 1 + len(L)

append : (List, Int) → List
append([], i) = i::[]
append(a :: L, i) = a :: append(L, i)

Claim: For all lists **L**, and integers **i**, if len(L) = n, then len(append(L, i)) = n + 1.

We go by structural induction on List. Let i be an integer, and let L be a list. Suppose len(L) = n. And Suppose len(append(L', i)) = k + 1 is true for some list L'.

Case L = $x :: L'$:

Recursively Defined Programs (on Lists)

len : List → Int
len([]) = 0
len(a :: L) = 1 + len(L)

append : (List, Int) → List
append([], i) = i::[]
append(a :: L, i) = a :: append(L, i)

Claim: For all lists **L**, and integers **i**, if len(L) = n, then len(append(L, i)) = n + 1.

We go by structural induction on List. Let i be an integer, and let L be a list. Suppose len(L) = n. And Suppose len(append(L', i)) = k + 1 is true for some list L'.

Case L = $x :: L'$:

len(append(x :: L', i))	= len(x :: append(L', i))	[Def of append]
	= 1 + len(append(L', i))	[Def of len]

We know by our IH that, for all lists smaller than L,
If len(L) = n, then len(append(L, i)) = n + 1

So, if len(L') = k, then len(append(L', i)) = k + 1

Recursively Defined Programs (on Lists)

We go by structural induction on List. Let i be an integer, and let L be a list. Suppose $\text{len}(\text{L}) = \text{n}$. And Suppose $\text{len}(\text{no1}(L')) = \#0(L')$ is true for some list L'.

Case L = $x :: L'$:

$\text{len}(\text{append}(\text{x} :: \text{L}', \text{i})) = \text{len}(\text{x} :: \text{append}(\text{L}', \text{i}))$ [Def of append]

$= 1 + \text{len}(\text{append}(\text{L}', \text{i}))$ [Def of len]

We know by our IH that, for all lists smaller than L,

If $\text{len}(\text{L}) = \text{n}$, then $\text{len}(\text{append}(\text{L}, \text{i})) = \text{n} + 1$

So, if $\text{len}(\text{L}') = \text{k}$, then $\text{len}(\text{append}(\text{L}', \text{i})) = \text{k} + 1$

$= 1 + \text{k} + 1$ [By IH]

Note that $\text{n} = \text{len}(\text{L}) = \text{len}(\text{x} :: \text{L}') = 1 + \text{len}(\text{L}') = 1 + \text{k}$.

$= 1 + (\text{n} - 1) + 1$ [By above]

$= \text{n} + 1$ [By above]