CSE 311

Foundations of Computing I

Administrivia

Token verifications will be e-mailed to you tonight!

The midterm will be on Wed, May 3 from 4:00pm – 5:30pm in KNE 120.

If you cannot make this time, and you haven't already e-mailed me, you need to tell me **right after lecture**.

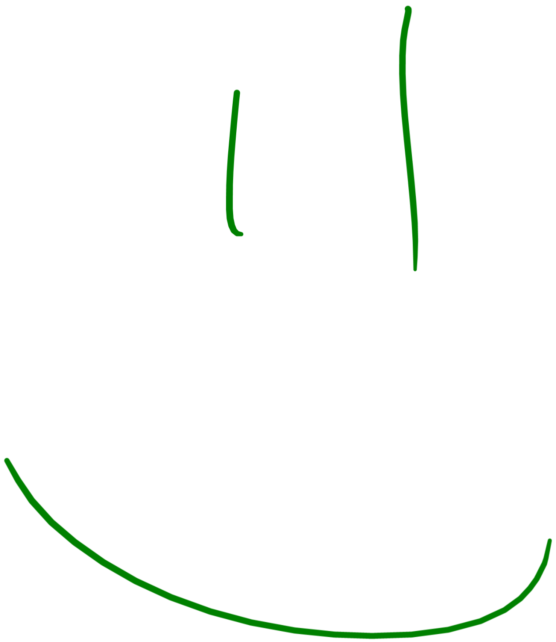
There will be two review sessions:

- Sunday from 12pm – 2pm in EEB 105
- Tuesday from 2:30pm – 4:30pm Location TBD

HW5-0 due tonight!

Dominos?

P(n) "n



Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$". We go by induction on n.

Base Case (n=0): Note that $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$, which is exactly $P(0)$.

Induction Hypothesis: Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step: We want to show $P(k+1)$. That is, we want to show: $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$

Note that $\sum_{i=0}^{k+1} 2^i = \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1}$ [Splitting the summation]

$= \left(2^{k+1} - 1\right) + 2^{k+1}$ [By IH]

$= \left(2^{k+1} + 2^{k+1}\right) - 1$ [Assoc. of +]

$= \left(2\left(2^{k+1}\right)\right) - 1$ [Factoring]

$= 2^{k+2} - 1$ [Simplifying]

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.

So, the claim is true for all natural numbers by induction.

Don't bother justifying the "obvious" steps. But make sure you say "by IH" somewhere.

We know (by IH)...

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$$

We're trying to get...

$$\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$$

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

Prove 1 + 2 + 3 + ... + n = n(n+1)/2

Let $P(n)$ be "$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$". We go by induction on n.

Base Case (n=0): $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$

Induction Hypothesis: Suppose $P(k)$ is true for some $k \in \mathbb{N}$

Induction Step:

$$\sum_{i=0}^{k+1} i = \sum_{i=0}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \quad \text{by IH}$$

$$= \frac{(k+1)(k+2)}{2}$$

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.

So, the claim is true for all natural numbers by induction.

We know (by IH)...

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$$

We're trying to get...

$$\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

Prove 1 + 2 + 3 + … + n = n(n+1)/2

Let $P(n)$ be "$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$". We go by induction on n.

Base Case (n=0): Note that $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$, which is exactly $P(0)$.

Induction Hypothesis: Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step: We want to show $P(k+1)$. That is, we want to show: $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$$\text{Note that } \sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i\right) + (k+1)$$ **[Splitting the summation]**

$$= \left(\frac{k(k+1)}{2}\right) + (k+1)$$ **[By IH]**

$$= (k+1)\left(\frac{k}{2} + 1\right) = (k+1)\left(\frac{k+2}{2}\right)$$ **[Algebra]**

$$= \frac{(k+1)(k+2)}{2}$$ **[Algebra]**

This is exactly $P(k+1)$. So, $P(k) \to P(k+1)$.

So, the claim is true for all natural numbers by induction.

We know (by IH)…

$$\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$$

We're trying to get…

$$\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

Prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

Let $P(n)$ be "$3 \mid 2^{2n} - 1$." We go by induction on n.

Base Case (n=0): $2^{2\cdot 0} - 1 = 2^0 - 1 = 0 = 3 \cdot 0$. So, $3 \mid 2^{2\cdot 0} - 1$ by def. of $\mid$.

Induction Hypothesis: Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step:

$$2^{2(k+1)} - 1 = 2^{2k}(2^2) - 1$$
$$= (2^{2k} - 1 + 1)\,2^2 - 1$$
$$= (3m + 1)\,2^2 - 1 \quad \text{for some } m \in \mathbb{Z} \text{ by IH.}$$
$$= 4 \cdot 3 \cdot m + 4 - 1$$
$$= 3(4m + 1)$$

$$= 3(\quad\quad)$$

We know (by IH)...

$3 \mid 2^{2k} - 1$

...which means...

$2^{2k} - 1 = 3m$

We're trying to get...

$3 \mid 2^{2(k+1)} - 1$

...which is true if...

$2^{2(k+1)} - 1 = 3q$

Prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

Let $P(n)$ be "$3 \mid 2^{2n} - 1$". We go by induction on n.

Base Case (n=0): Note that $2^{2 \cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$.

We know $3 \mid 0$, by definition of divides, because $3 \cdot 0 = 0$. So, P(0) is true.

Induction Hypothesis: Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step: We want to show $P(k+1)$. That is, WTS $3 \mid 2^{2(k+1)} - 1$.

$$\text{Note that } 2^{2(k+1)} - 1 = 2^{2k+2} - 1 \quad \text{[Algebra]}$$

$$= (2^{2k})(2^2) - 1 \quad \text{[Algebra]}$$

$$= (2^{2k} - 1 + 1)(2^2) - 1 \quad \text{[Algebra]}$$

By IH, we know $3 \mid 2^{2k} - 1$. So, by definition of divides, we know $2^{2k} - 1 = 3j$ for some j.

$$= (3j + 1)(4) - 1 = 3(4j + 1) \quad \text{[Algebra]}$$

So, by definition of divides, $3 \mid 2^{2(k+1)} - 1$.

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.

So, the claim is true for all natural numbers by induction.

We know (by IH)...

$3 \mid 2^{2k} - 1$

...which means...

$2^{2k} - 1 = 3j$

We're trying to get...

$3 \mid 2^{2(k+1)} - 1$

...which is true if...

$2^{2(k+1)} - 1 = 3k$

Prove $3^n \geq n^2$ for all $n \geq 3$.

Let $P(n)$ be "$3^n \geq n^2$". We go by induction on n.

Base Case (n=3):

Induction Hypothesis:

Induction Step: We want to show $P(k+1)$.

Note that

We know (by IH)…

We're trying to get…

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.

So, the claim is true for all $n \geq 3$ by induction.

Prove $3^n \geq n^2$ for all $n \geq 3$.

Let $P(n)$ be "$3^n \geq n^2$". We go by induction on n.

Base Case (n=3): Note that $3^3 = 27 \geq 9 = 3^2$. So, P(3) is true.

Induction Hypothesis: Suppose $P(k)$ is true for some $k \geq 3$.

Induction Step: We want to show $P(k+1)$.

Note that $3^{k+1} = 3(3^k)$ [Algebra]

$\geq 3(k^2)$ [By IH]

$= k^2 + k \cdot k + k^2$ [Algebra]

$\geq k^2 + 2 \cdot k + k^2$ [k ≥ 2]

$\geq k^2 + 2 \cdot k + 1^2$ [k ≥ 1]

$\geq k^2 + 2k + 1$

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.

So, the claim is true for all $n \geq 3$ by induction.

We know (by IH)...

$3^k \geq k^2$

We're trying to get...

$3^{k+1} \geq (k+1)^2$

$= k^2 + 2k + 1$

Prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Let $P(n)$ be "$2n^3 + 2n - 5 \geq n^2$". We go by induction on n.

Base Case (n=2):

Induction Hypothesis:

Induction S[illegible] We want to show $P(k+1)$.

For Later!!!!

We know (by IH)...

We're trying to get...

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.

So, the claim is true for all $n \geq 2$ by induction.

Prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Let $P(n)$ be "$2n^3 + 2n - 5 \geq n^2$". We go by induction on n.

Base Case (n=2): Note that $2(2^3) + 2(2) - 5 = 15 \geq 4 = 2^2$

Induction Hypothesis: Suppose the claim is true for some $k \geq 2$.

Induction Step: We want to show $P(k+1)$.

Note that $2(k+1)^3 + (2k+1) - 5 = 2(k+1)(k^2 + 2k + 1) + (2k+1) - 5$

[Algebra]

$$= 2(k^3 + 2k^2 + k + k^2 + 2k + 1) + (2k+1) - 5$$
$$= 2k^3 + 4k^2 + 2k + 2k^2 + 4k + 2 + (2k+1) - 5$$
$$= 2k^3 + 6k^2 + 6k + 2 + (2k+1) - 5$$
$$= (2k^3 + 2k - 5) + 6k^2 + 6k + 3$$

[By IH] $\geq k^2 + 6k^2 + 6k + 3 = 7k^2 + 6k + 3$

[Algebra]

$$= (k^2 + 2k + 1) + 6k^2 + 4k + 3$$
$$= (k+1)^2 + 6k^2 + 4k + 3$$

[k ≥ 2] $\geq (k+1)^2$

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.

So, the claim is true for all $n \geq 2$ by induction.

We know (by IH)...

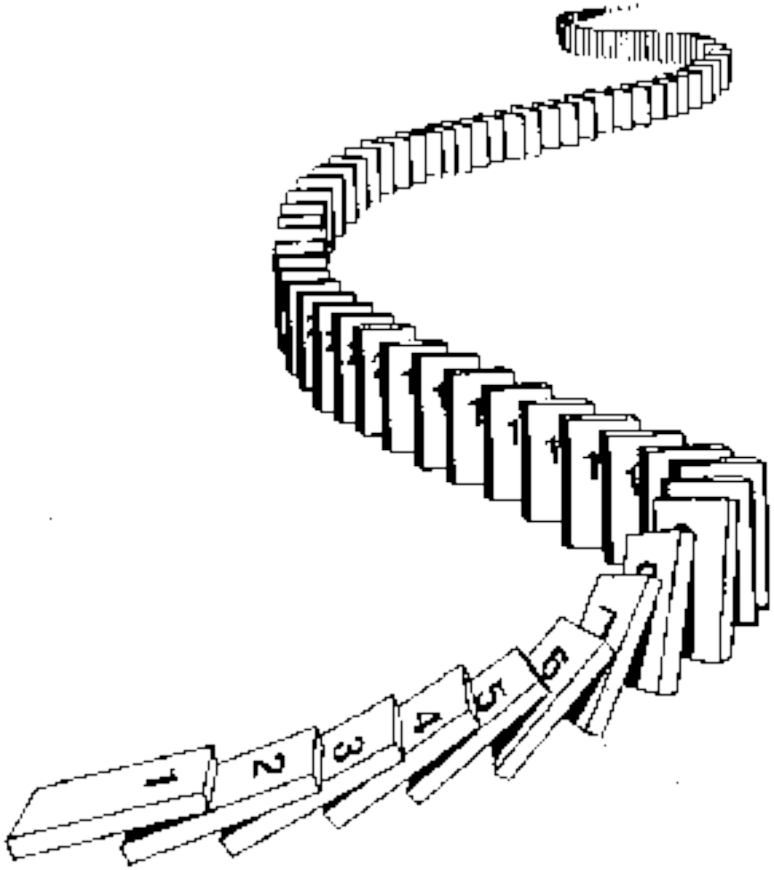
$$2k^3 + 2k - 5 \geq k^2$$

We're trying to get...

$$2(k+1)^3 + 2(k+1) - 5 \geq (k+1)^2$$
$$(k+1)^2 = k^2 + 2k + 1$$

CSE 311: Foundations of Computing

Lecture 15: Strong Induction



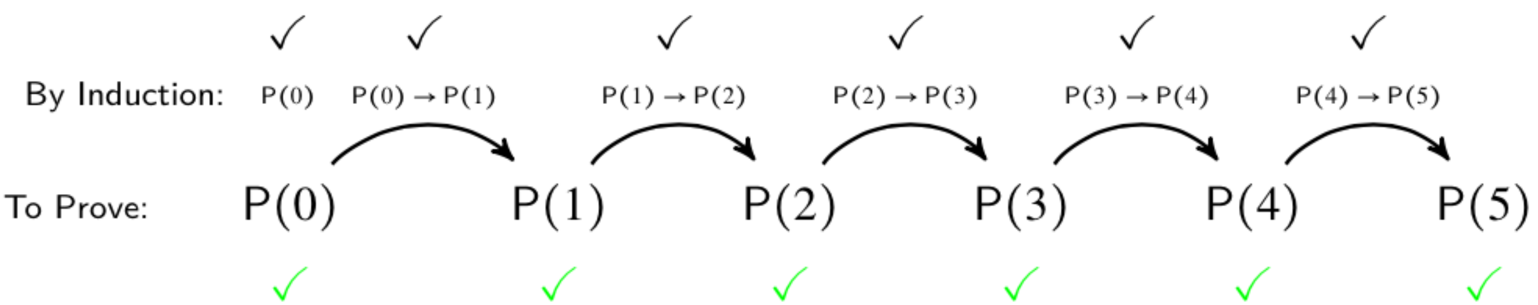
Induction Is A Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall\, k\, (P(k) \rightarrow P(k+1))$$

$$\therefore\ \forall\, n\, P(n)$$

How does this technique prove P(5)?



First, we prove P(0).

Since P(n) → P(n+1) for all n, we have P(0) → P(1).

Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.

Since P(n) → P(n+1) for all n, we have P(1) → P(2).

Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Domain: Natural Numbers

Induction Is A Rule of Inference...Again

1.	P(0)	("Given")
2.	$\forall n\ (\mathrm{P}(n) \rightarrow \mathrm{P}(n+1))$	("Given")
3.	P(1)	(MP: 2, 1)
4.	P(2)	(MP: 2, 3)
5.	P(3)	(MP: 2, 4)
6.	P(4)	(MP: 2, 5)

Induction Is A Rule of Inference

"Induction"

1.	P(0)	("Given")
2.	$\forall n\ (P(n) \rightarrow P(n+1))$	("Given")
3.	P(1)	(MP: 2, 1)
4.	P(2)	(MP: 2, 3)
5.	P(3)	(MP: 2, 4)
6.	P(4)	(MP: 2, 5)

Notice how when we use regular induction, we're already proving the things necessary to use strong induction.

This is no extra work with a benefit!

"Strong Induction"

1.	P(0)	("Given")
2.	$\forall n\ ((P(0) \wedge P(1) \wedge \cdots \wedge P(n) \rightarrow P(n+1))$	("Given")
3.	P(1)	(MP: 2, 1)
4.	P(2)	(MP: 2, 1, 3)
5.	P(3)	(MP: 2, 1, 3, 4)
6.	P(4)	(MP: 2, 1, 3, 4, 5)

Strong Induction

$$P(0)$$

$$\forall k \left(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k) \big) \to P(k+1) \right)$$

$$\therefore \forall n \, P(n)$$

Strong Induction English Proof

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

Every $n \geq 2$ can be expressed as a product of primes.

Let $P(n)$ be "$n = p_0 p_1 \cdots p_j$, where $p_0, p_1, \ldots, p_j$ are prime."

We go by strong induction on n.

Base Case (n=2):

Induction Hypothesis:

Induction Step: We go by cases.

We know (by IH)…

All numbers smaller than k can be expressed as a product of primes.

We're trying to get…

k can be expressed as a product of primes.

Every $n \geq 2$ can be expressed as a product of primes.

Let $P(n)$ be "$n = p_0 p_1 \cdots p_j$, where $p_0, p_1, \ldots, p_j$ are prime."

We go by induction on n.

Base Case (n=2): Note that 2 is prime (which means it's a product of primes).

Induction Hypothesis: Suppose that P(2), P(3), …, P(k – 1) are true for some $k \geq 2$.

Induction Step: We go by cases.

Case 1 (k is prime):

Then, since k is prime, k is a product of primes.

Case 2 (k is composite):

Then, by definition of composite, we have non-trivial $1 < a, b < k$ such that k = ab. Since a and b are between 2 and k – 1, we know P(2) and P(k – 1) are true. So, we have:

$$a = p_0 p_1 \cdots p_j \text{ and } b = p_{j+1} p_{j+2} \cdots p_{j+\ell}$$

Then, k = ab = $p_0 p_1 \cdots p_j p_{j+1} p_{j+2} \cdots p_{j+\ell}$

So, k can be expressed as a product of primes.

So, P(n) is true for all $n \geq 2$ is true by induction.

We know (by IH)…

All numbers smaller than k can be expressed as a product of primes.

We're trying to get…

k can be expressed as a product of primes.