



# Foundations of Computing I

## Administrivia

Token verifications will be e-mailed to you tonight!

The midterm will be on Wed, May 3 from 4:00pm – 5:30pm in KNE 120.

If you cannot make this time, and you haven't already e-mailed me, you need to tell me **right after lecture**.

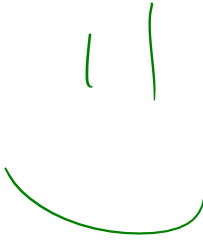
There will be two review sessions:

- Sunday from 12pm – 2pm in EEB 105
- Tuesday from 2:30pm – 4:30pm Location TBD

*AWS-0 due tonight!*

## Dominos?

*P(n) "is"*



## Prove $1 + 2 + 4 + \dots + 2^n = 2^{n+1} - 1$

Let  $P(n)$  be  $\sum_{i=0}^n 2^i = 2^{n+1} - 1$ . We go by induction on  $n$ .

**Base Case (n=0):** Note that  $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ , which is exactly  $P(0)$ .

**Induction Hypothesis:** Suppose  $P(k)$  is true for some  $k \in \mathbb{N}$ .

**Induction Step:** We want to show  $P(k+1)$ . That is, we want to show:  $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$

Note that  $\sum_{i=0}^{k+1} 2^i = \left( \sum_{i=0}^k 2^i \right) + 2^{k+1}$  [Splitting the summation]

$= (2^{k+1} - 1) + 2^{k+1}$  [By IH]

Don't bother justifying the "obvious" steps. But make sure you say "by IH" somewhere.

$= (2^{k+1} + 2^{k+1}) - 1$  [Assoc. of +]

$= (2(2^{k+1})) - 1$  [Factoring]

$= 2^{k+2} - 1$  [Simplifying]

This is exactly  $P(k+1)$ . So,  $P(k) \rightarrow P(k+1)$ .

So, the claim is true for all natural numbers by induction.

We know (by IH)...

$$\sum_{i=0}^k 2^i = 2^{k+1} - 1$$

We're trying to get...

$$\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$$

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

## Prove $1 + 2 + 3 + \dots + n = n(n+1)/2$

Let  $P(n)$  be  $\sum_{i=0}^n i = \frac{n(n+1)}{2}$ . We go by induction on  $n$ .

**Base Case (n=0):**  $\sum_{i=0}^0 i = 0 = \frac{0(0+1)}{2}$

**Induction Hypothesis:** Suppose  $P(k)$  is true for some  $k \in \mathbb{N}$ .

**Induction Step:**

$$\sum_{i=0}^{k+1} i = \sum_{i=0}^k i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \quad \text{by IH}$$

$$= \frac{(k+1)(k+2)}{2}$$

We know (by IH)...

$$\sum_{i=0}^k i = \frac{k(k+1)}{2}$$

We're trying to get...

$$\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

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Note that  $\sum_{i=0}^{k+1} i = \left( \sum_{i=0}^k i \right) + (k+1)$  [Splitting the summation]

$= \left( \frac{k(k+1)}{2} \right) + (k+1)$  [By IH]

$= (k+1) \left( \frac{k}{2} + 1 \right) = (k+1) \left( \frac{k+2}{2} \right)$  [Algebra]

$= \frac{(k+1)(k+2)}{2}$  [Algebra]

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So, the claim is true for all natural numbers by induction.

We know (by IH)...

$$\sum_{i=0}^k i = \frac{k(k+1)}{2}$$

We're trying to get...

$$\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

### Prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$ .

Let  $P(n)$  be " $3 \mid 2^{2n} - 1$ ". We go by induction on  $n$ .

Base Case (n=0):  $2^{2 \cdot 0} - 1 = 2^0 - 1 = 0 = 3 \cdot 0$ . So,  $3 \mid 2^{2 \cdot 0} - 1$  by def. or 1.

Induction Hypothesis: Suppose  $P(k)$  is true for some  $k \in \mathbb{N}$ .

Induction Step:

$$\begin{aligned} 2^{2(k+1)} - 1 &= 2^{2k} (2^2) - 1 \\ &= (2^{2k} - 1 + 1) 2^2 - 1 \\ &= (3m + 1) 2^2 - 1 \quad \text{for some } m \in \mathbb{Z} \text{ by IH.} \\ &= 4 \cdot 3m + 4 - 1 \\ &= 3(4m + 1) \\ &= 3(\dots) \end{aligned}$$

We know (by IH)...

$$3 \mid 2^{2k} - 1$$

...which means...

$$2^{2k} - 1 = 3m$$

We're trying to get...

$$3 \mid 2^{2(k+1)} - 1$$

...which is true if...

$$2^{2(k+1)} - 1 = 3l$$

### Prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$ .

Let  $P(n)$  be " $3 \mid 2^{2n} - 1$ ". We go by induction on  $n$ .

Base Case (n=0): Note that  $2^{2 \cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$ .

We know  $3 \mid 0$ , by definition of divides, because  $3 \cdot 0 = 0$ . So,  $P(0)$  is true.

Induction Hypothesis: Suppose  $P(k)$  is true for some  $k \in \mathbb{N}$ .

Induction Step: We want to show  $P(k+1)$ . That is, WTS  $3 \mid 2^{2(k+1)} - 1$ .

Note that  $2^{2(k+1)} - 1 = 2^{2k+2} - 1$

[Algebra]

$$= (2^{2k})(2^2) - 1$$

[Algebra]

$$= (2^{2k} - 1 + 1)(2^2) - 1$$

[Algebra]

By IH, we know  $3 \mid 2^{2k} - 1$ . So, by definition of divides, we know  $2^{2k} - 1 = 3j$  for some  $j$ .

$$= (3j + 1)(4) - 1 = 3(4j + 1)$$

[Algebra]

So, by definition of divides,  $3 \mid 2^{2(k+1)} - 1$ .

This is exactly  $P(k+1)$ . So,  $P(k) \rightarrow P(k+1)$ .

So, the claim is true for all natural numbers by induction.

We know (by IH)...

$$3 \mid 2^{2k} - 1$$

...which means...

$$2^{2k} - 1 = 3j$$

We're trying to get...

$$3 \mid 2^{2(k+1)} - 1$$

...which is true if...

$$2^{2(k+1)} - 1 = 3k$$

### Prove $3^n \geq n^2$ for all $n \geq 3$ .

Let  $P(n)$  be " $3^n \geq n^2$ ". We go by induction on  $n$ .

Base Case (n=3):

Induction Hypothesis:

Induction Step: We want to show  $P(k+1)$ .

Note that

We know (by IH)...

We're trying to get...

This is exactly  $P(k+1)$ . So,  $P(k) \rightarrow P(k+1)$ .

So, the claim is true for all  $n \geq 3$  by induction.

### Prove $3^n \geq n^2$ for all $n \geq 3$ .

Let  $P(n)$  be " $3^n \geq n^2$ ". We go by induction on  $n$ .

Base Case (n=3): Note that  $3^3 = 27 \geq 9 = 3^2$ . So,  $P(3)$  is true.

Induction Hypothesis: Suppose  $P(k)$  is true for some  $k \geq 3$ .

Induction Step: We want to show  $P(k+1)$ .

Note that  $3^{k+1} = 3(3^k)$

[Algebra]

$$\geq 3(k^2)$$

[By IH]

$$= k^2 + k \cdot k + k^2$$

[Algebra]

$$\geq k^2 + 2 \cdot k + k^2 \quad [k \geq 2]$$

$$\geq k^2 + 2 \cdot k + 1^2 \quad [k \geq 1]$$

$$\geq k^2 + 2k + 1$$

We know (by IH)...

$$3^k \geq k^2$$

We're trying to get...

$$3^{k+1} \geq (k+1)^2$$

$$= k^2 + 2k + 1$$

This is exactly  $P(k+1)$ . So,  $P(k) \rightarrow P(k+1)$ .

So, the claim is true for all  $n \geq 3$  by induction.

### Prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$ .

Let  $P(n)$  be " $2n^3 + 2n - 5 \geq n^2$ ". We go by induction on  $n$ .

Base Case (n=2):

Induction Hypothesis:

Induction Step: We want to show  $P(k+1)$ .

For Later!!!!

We know (by IH)...

We're trying to get...

This is exactly  $P(k+1)$ . So,  $P(k) \rightarrow P(k+1)$ .

So, the claim is true for all  $n \geq 2$  by induction.

### Prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$ .

Let  $P(n)$  be " $2n^3 + 2n - 5 \geq n^2$ ". We go by induction on  $n$ .

Base Case (n=2): Note that  $2(2^3) + 2(2) - 5 = 15 \geq 4 = 2^2$

Induction Hypothesis: Suppose the claim is true for some  $k \geq 2$ .

Induction Step: We want to show  $P(k+1)$ .

Note that  $2(k+1)^3 + 2(k+1) - 5 = 2(k+1)(k^2 + 2k + 1) + (2k+1) - 5$

$$\begin{aligned} &= 2(k^3 + 2k^2 + k + k^2 + 2k + 1) + (2k+1) - 5 \\ &= 2k^3 + 4k^2 + 2k + 2k^2 + 4k + 2 + (2k+1) - 5 \\ &= 2k^3 + 6k^2 + 6k + 2 + (2k+1) - 5 \\ &= (2k^3 + 2k - 5) + 6k^2 + 6k + 3 \end{aligned}$$

[By IH]

$$\geq k^2 + 6k^2 + 6k + 3 = 7k^2 + 6k + 3$$

$$= (k^2 + 2k + 1) + 6k^2 + 4k + 3$$

[Algebra]

$$= (k+1)^2 + 6k^2 + 4k + 3$$

$$[k \geq 2] \geq (k+1)^2$$

We know (by IH)...

$$2k^3 + 2k - 5 \geq k^2$$

We're trying to get...

$$2(k+1)^3 + 2(k+1) - 5 \geq (k+1)^2$$

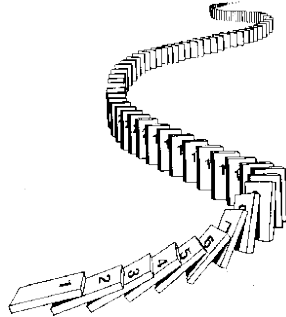
$$(k+1)^2 = k^2 + 2k + 1$$

This is exactly  $P(k+1)$ . So,  $P(k) \rightarrow P(k+1)$ .

So, the claim is true for all  $n \geq 2$  by induction.

## CSE 311: Foundations of Computing

### Lecture 15: Strong Induction

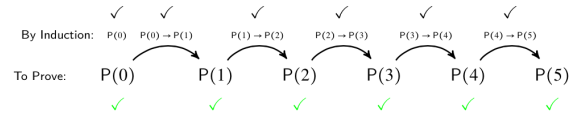


## Induction Is A Rule of Inference

Domain: Natural Numbers  $P(0)$   
 $\forall k (P(k) \rightarrow P(k+1))$

$\therefore \forall n P(n)$

How does this technique prove  $P(5)$ ?



First, we prove  $P(0)$ .

Since  $P(n) \rightarrow P(n+1)$  for all  $n$ , we have  $P(0) \rightarrow P(1)$ .

Since  $P(0)$  is true and  $P(0) \rightarrow P(1)$ , by Modus Ponens,  $P(1)$  is true.

Since  $P(n) \rightarrow P(n+1)$  for all  $n$ , we have  $P(1) \rightarrow P(2)$ .

Since  $P(1)$  is true and  $P(1) \rightarrow P(2)$ , by Modus Ponens,  $P(2)$  is true.

## Induction Is A Rule of Inference...Again

Domain: Natural Numbers

1.  $P(0)$  ("Given")
2.  $\forall n (P(n) \rightarrow P(n+1))$  ("Given")
3.  $P(1)$  (MP: 2, 1)
4.  $P(2)$  (MP: 2, 3)
5.  $P(3)$  (MP: 2, 4)
6.  $P(4)$  (MP: 2, 5)

## Induction Is A Rule of Inference

Domain: Natural Numbers

"Induction"

1.  $P(0)$  ("Given")
2.  $\forall n (P(n) \rightarrow P(n+1))$  ("Given")
3.  $P(1)$  (MP: 2, 1)
4.  $P(2)$  (MP: 2, 3)
5.  $P(3)$  (MP: 2, 4)
6.  $P(4)$  (MP: 2, 5)

Notice how when we use regular induction, we're already proving the things necessary to use strong induction.

This is no extra work with a benefit!

"Strong Induction"

1.  $P(0)$  ("Given")
2.  $\forall n ((P(0) \wedge P(1) \wedge \dots \wedge P(n)) \rightarrow P(n+1))$  ("Given")
3.  $P(1)$  (MP: 2, 1)
4.  $P(2)$  (MP: 2, 1, 3)
5.  $P(3)$  (MP: 2, 1, 3, 4)
6.  $P(4)$  (MP: 2, 1, 3, 4, 5)

## Strong Induction

$P(0)$   
 $\forall k ((P(0) \wedge P(1) \wedge P(2) \wedge \dots \wedge P(k)) \rightarrow P(k+1))$

$\therefore \forall n P(n)$

## Strong Induction English Proof

1. By induction we will show that  $P(n)$  is true for every  $n \geq 0$
2. Base Case: Prove  $P(0)$
3. Inductive Hypothesis:  
Assume that for some arbitrary integer  $k \geq 0$ ,  $P(j)$  is true for every  $j$  from 0 to  $k$
4. Inductive Step:  
Prove that  $P(k+1)$  is true using the Inductive Hypothesis (that  $P(j)$  is true for all values  $\leq k$ )
5. Conclusion: Result follows by induction

**Every  $n \geq 2$  can be expressed as a product of primes.**

Let  $P(n)$  be " $n = p_0 p_1 \cdots p_j$ , where  $p_0, p_1, \dots, p_j$  are prime."

We go by strong induction on  $n$ .

Base Case ( $n=2$ ):

Induction Hypothesis:

Induction Step: We go by cases.

We know (by IH)...

All numbers smaller than  $k$  can be expressed as a product of primes.

We're trying to get...

$k$  can be expressed as a product of primes.

**Every  $n \geq 2$  can be expressed as a product of primes.**

Let  $P(n)$  be " $n = p_0 p_1 \cdots p_j$ , where  $p_0, p_1, \dots, p_j$  are prime."

We go by induction on  $n$ .

Base Case ( $n=2$ ): Note that 2 is prime (which means it's a product of primes).

Induction Hypothesis: Suppose that  $P(2), P(3), \dots, P(k - 1)$  are true for some  $k \geq 2$ .

Induction Step: We go by cases.

Case 1 ( $k$  is prime):

Then, since  $k$  is prime,  $k$  is a product of primes.

Case 2 ( $k$  is composite):

Then, by definition of composite, we have non-trivial

$1 < a, b < k$  such that  $k = ab$ . Since  $a$  and  $b$  are between 2

and  $k - 1$ , we know  $P(2)$  and  $P(k - 1)$  are true. So, we have:

$$a = p_0 p_1 \cdots p_j \text{ and } b = p_{j+1} p_{j+2} \cdots p_{j+\ell}$$

Then,  $k = ab = p_0 p_1 \cdots p_j p_{j+1} p_{j+2} \cdots p_{j+\ell}$

So,  $k$  can be expressed as a product of primes.

So,  $P(n)$  is true for all  $n \geq 2$  is true by induction.

We know (by IH)...

All numbers smaller than  $k$  can be expressed as a product of primes.

We're trying to get...

$k$  can be expressed as a product of primes.