CSE 311: Foundations of Computing I

Understanding Implications

The Definition

We define $p \rightarrow q$ ("p implies q", "if p, then q") by the following truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

This means that **an implication is only false when p is true and q is false**. This leads to a useful way of thinking about implications:

The implication $p \rightarrow q$ is a *promise* that whenever p is true, q is also true.

Similarly, the implication is only false when *the promise is broken*.

Translating Implication

We know the truth table for implication. So, we can read the true cases off of it for the statement $p \rightarrow q$:

$$p = \mathsf{T}, q = \mathsf{T} \qquad\qquad p = \mathsf{F}, q = \mathsf{T} \qquad\qquad p = \mathsf{F}, q = \mathsf{F}$$

In **any** of these situations, the implication is true. So, putting this together, we have:

$$(p = \mathsf{T} \land q = \mathsf{T}) \lor (p = \mathsf{F} \land q = \mathsf{T}) \lor (p = \mathsf{F} \land q = \mathsf{F})$$

We can't actually use "equals" in our expressions, but asserting "p" is the same as insisting $p = \mathsf{T}$, and asserting $\neg p$ is the same as insisting $p = \mathsf{F}$. So, using these ideas:

$$(p \land q) \lor (\neg p \land q) \lor (\neg p \land \neg q)$$

But we can simplify this. Notice that we can factor out a q from the first two:

$$((p \lor \neg p) \land q) \lor (\neg p \land \neg q)$$

Then, getting rid of $p \lor \neg p$, we have

$$q \lor (\neg p \land \neg q)$$

Distributing out the q, we have:

$$(\neg p \lor q) \land (q \lor \neg q)$$

Getting rid of $q \lor \neg q$, we're left with:

$$\neg p \lor q$$

So, $p \rightarrow q \equiv \neg p \lor q$.

Implication is not Causal!

It's tempting to think of "if p, then q" statements as meaning "p is the cause of q", but this is not what implication means. Consider the following:

Let p be the proposition "$2 + 2 = 4$", and let q be the proposition "Today is Monday". It should be clear that the day of the week and the sum $2 + 2$ have nothing to do with each other. Nevertheless, the implication *is true* when it's Monday. It's important to realize that implication works on the level of *truth values*. The statements really don't matter–only when they're true matters!

If, Only If, and If and only if

The implications $p \to q$ and $q \to p$ do not always have the same truth value. For example, consider p as the proposition "Today is Monday" and q as the proposition "Today is not Wednesday". Clearly, $p \to q$ is true, because it's only one day at a time. But, the reverse implication is *not always true*. If it's Thursday, it is not Wednesday (so p is true), but it's also not Monday (so q is false); this makes the implication false.

In English, we call one of these directions "p if q" and the other "p only if q". Here's how to think about it:

- p if q translates directly into "if q, then p". We know this is $q \to p$. Let's develop some intuition though. Imagine I assert:

  "I have my umbrella **if** it's raining"

  As always, when would this be a lie? When I don't have my umbrella and it's raining, then I'm lying. That means this *isn't a lie* when I have my umbrella or it's not raining. Notice that this is exactly the expression we got for "if it's raining, then I have my umbrella".

- p only if q must be the other direction "if p, then q", which is $p \to q$. But, again, why? Imagine I assert:

  "I have my umbrella **only if** it's raining"

  This would be a lie in the case where it's not raining and I have my umbrella, right? So, it wouldn't be a lie in the case where it's raining or I don't have my umbrella. That's "if I have my umbrella, then it's raining".

"p if and only if q" (or p iff q) is just the assertion that both $p \to q$ and $q \to p$ are true.

You might also come across "Unless p, q". Again, let's go back to our example:

"**Unless** it's raining, I have my umbrella "

The easiest way to handle this statement is to translate *unless* as *if not*. So, this becomes $\neg p \to q$.

Note that there is a "stronger"(meaning, it effects more situations) definition of unless that leads to a different answer which we will not discuss.