Adam Blank Spring 2017

# CSE 311

# Foundations of Computing I

## Pre-Lecture Problem

Translate the following into predicate logic:

"There is a student who is friends with every other student except her enemies."

Hi!

## Negations of Quantifiers

Domain: Fruit

| Predicate Definitions |
| --- |
| PF(x) ::= "x is a purple fruit" |

$\forall x\ PF(x)$

Imagine our domain is {plum, banana, apple}.
Can you write the statement without any quantifiers?

$\neg(PF(plum) \wedge PF(banana) \wedge PF(apple))$

What is the negation of that statement?

$\neg PF(plum) \vee \neg PF(banana) \vee \neg PF(apple)$

$\exists x\ \neg P(x)$

## Negations of Quantifiers

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$PF(plum) \wedge PF(banana) \wedge PF(apple)$

What is the negation of that statement?

$\neg(PF(plum) \wedge PF(banana) \wedge PF(apple))$
$\equiv \neg PF(plum) \vee \neg PF(banana) \vee \neg PF(apple)$

"One of the fruits is not purple"

$\exists x\ \neg P(x)$

## De Morgan's Laws for Quantifiers

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$$
$$\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

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$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$$
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"There is no largest integer"

$$\forall x\ \big(\neg(\forall y(x \geq y))\big) \equiv \forall x\ \big(\exists y(\neg(x \geq y))\big)$$
$$\equiv \forall x\ \big(\exists y\ (x < y)\big)$$

"For every integer there is a larger integer"

## Negations of Quantifiers

- not every positive integer is prime
- some positive integer is not prime
- prime numbers do not exist
- every positive integer is not prime

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- not every positive integer is prime
  $\neg\forall x\ \text{Prime}(x) \equiv \exists x\ \neg\text{Prime}(x)$
- some positive integer is not prime
  $\exists x\ \neg\text{Prime}(x)$
- prime numbers do not exist
  $\neg\exists x\ \text{Prime}(x)$
- every positive integer is not prime
  $\forall x \neg\text{Prime}(x)$

## Bound and Free Variables

Consider the following program:

```
hello(x) {
    return x + y;
}
```

In this program, we say "x" is bound and "y" is free.

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Consider the following program:

```
hello(x) {
    return x + y;
}
```

x is defined here

x is used here

y is never defined ☹

In this program, we say "x" is bound and "y" is free.

## Scope of Quantifiers

It's the same idea with quantifiers.

$\exists y$ Greater(y, x)

$\forall x\ \exists y$ Greater(y, x)

We figure out what a formula means "inside-out".

So, variables bind to the inner-most quantifier that tries to "capture" them.

$\forall x\ \exists y$ Greater(y, x)

This quantifier does nothing!

$\forall x\ (\exists y\ (P(x,y) \rightarrow \forall x\ Q(y, x)))$

## BTW: Quantifier "Style"

$\forall x\ \exists y\ (P(x,y) \rightarrow \forall x\ Q(y, x)))$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…

## Scope of Quantifiers

$\exists x\,(P(x) \wedge Q(x))$ **vs.** $\exists x\,P(x) \wedge \exists x\,Q(x)$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

## Variable Renaming

NotLargest1(x) ≡ ∃y Greater(y, x) **vs.** NotLargest2(x) ≡ ∃z Greater(z, x)

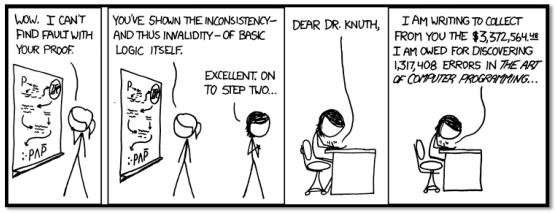
```
notLargest1(x) {
    boolean result = false;
    for(y : DOMAIN) {
        result = result || y > x
    }
    return result;
}
```

```
notLargest2(x) {
    boolean result = false;
    for(z : DOMAIN) {
        result = result || z > x
    }
    return result;
}
```

These are the same program!
Variable names are irrelevant!

## CSE 311: Foundations of Computing

### Lecture 6: Predicate Logic, Logical Inference




## MMM Candy!

Let B be the "starred" bag of M&Ms.
Translate "There is a green M&M in B." into predicate logic.

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Translate "~~There is a green M&M in B.~~" into predicate logic.

| Domain of Discourse |
|---|
| Colors & Bags |

| Predicate Definitions |
|---|
| Bag(x) ::= "x is a bag of M&Ms" |
| Color(x) ::= "x is a color" |
| Has(b, c) ::= "b has a c M&M." |

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$\text{Has}(\textbf{Green}, \textbf{B})$

Notice that both "Green" and "B" are constants here! You could, instead, define a predicate Green(x):

$\exists c\big(\text{Color}(c) \wedge \text{Green}(c) \wedge \text{Has}(c, \textbf{B})\big)$

## MMM Candy!

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Translate "Every bag of M&Ms has an M&M of some color."

$\forall x \left(\text{Bag}(x) \rightarrow \exists c \left(\text{Color}(c) \wedge \text{Has}(x, c)\right)\right)$

Translate "There is a color that all bags of M&Ms have."

$\exists c \left(\text{Color}(c) \wedge \forall b \left(\text{Bag}(b) \rightarrow \text{Has}(b, c)\right)\right)$

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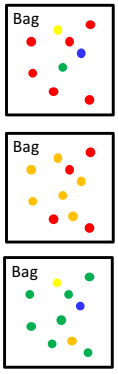


"There is a color that all bags of M&Ms have."

$\exists c \left(\text{Color}(c) \wedge \forall b \left(\text{Bag}(b) \rightarrow \text{Has}(b, c)\right)\right)$

**vs.**

"Every bag of M&Ms has an M&M of some color."

$\forall b \left(\text{Bag}(b) \rightarrow \exists c \left(\text{Color}(c) \wedge \text{Has}(b, c)\right)\right)$

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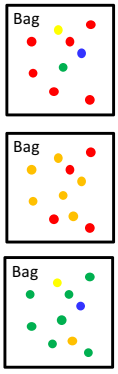


"There is a color that all bags of M&Ms have."

$\exists c \left(\text{Color}(c) \wedge \forall b \left(\text{Bag}(b) \rightarrow \text{Has}(b, c)\right)\right)$

For the bags on the left, this is not true. We need **a single color** that all the bags share.

**vs.**

"Every bag of M&Ms has an M&M of some color."

$\forall b \left(\text{Bag}(b) \rightarrow \exists c \left(\text{Color}(c) \wedge \text{Has}(b, c)\right)\right)$

For the bags on the left, this **is true**. The first bag has red; the second has orange, and the third has orange.

## Still not done!

| Domain of Discourse |
|---|
| {1, 2, 3, 4} |

| Predicate Definitions |
|---|
| GreaterEq(x, y) ::= "x ≥ y" |

"There is a number greater than or equal to all numbers."

$\exists x \left(\forall y \left(\text{GreaterEq}(x, y)\right)\right)$

**vs.**

"Every number has a number greater than or equal to it."

$\forall y \left(\exists x \left(\text{GreateEq}(x, y)\right)\right)$

## Still not done!

| Domain of Discourse |
|---|
| Integers<br>OR<br>{1, 2, 3, 4} |

| Predicate Definitions |
|---|
| GreaterEq(x, y) ::= "x ≥ y" |

"There is a number greater than or equal to all numbers."

$\exists x \left(\forall y \left(\text{GreaterEq}(x, y)\right)\right)$

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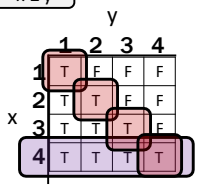
$\forall y \left(\exists x \left(\text{GreaterEq}(x, y)\right)\right)$




The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

## Nested Quantifiers

- Bound variable names don't matter

  $\forall x \exists y\ P(x, y) \equiv \forall a \exists b\ P(a, b)$

- Positions of quantifiers can sometimes change

  $\forall x\ (Q(x) \wedge \exists y\ P(x, y)) \equiv \forall x \exists y\ (Q(x) \wedge P(x, y))$

- But: order is important…

## Quantification with Two Variables

| expression | when true | when false |
|---|---|---|
| $\forall x \forall y\ P(x, y)$ | Every pair is true. | At least one pair is false. |
| $\exists x \exists y\ P(x, y)$ | At least one pair is true. | All pairs are false. |
| $\forall x \exists y\ P(x, y)$ | We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ | Some x doesn't have a corresponding y. |
| $\exists y \forall x\ P(x, y)$ | We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$ | For any candidate y, there is an x that it doesn't work for. |

## Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other
- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

## Why Proofs?

Consider f(n) = 991n^2 + 1

Is f(n) a perfect square for any n > 0?

## Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

## Proofs

- Start with hypotheses and facts (**Axioms**)
- Use "rules" to generate more facts from existing facts (**Inference Rules**)
- Result is proved when it is included in the set of "proven facts"

## Inference Rules

If I have a proof of A and I have a proof of B....

Requirements: A B

Conclusions: ∴ C D

Then, I have also proven C

Then, I have also proven D

Example (Modus Ponens):

$$\frac{A \quad A \rightarrow B}{\therefore \quad B}$$

If I have a proof of A and a proof of A → B, then I have a proof of B.

## An inference rule: *Modus Ponens*

- If p and p → q are both true then q must be true
- Write this rule as

| Modus Ponens | |
|---|---|
| A | A → B |
| ∴ | B |

- Given:
  - If it's Saturday, then you have a 311 lecture today.
  - It's Saturday.
- Therefore, by modus ponens:
  - You have a 311 lecture today.

## My First Proof!

Show that r follows from p, p → q, and q → r

| | | |
|---|---|---|
| 1. | p | Given |
| 2. | p → q | Given |
| 3. | q → r | Given |
| 4. | | |
| 5. | | |

## My First Proof!

Show that r follows from p, p → q, and q → r

| | | |
|---|---|---|
| 1. | p | Given |
| 2. | p → q | Given |
| 3. | q → r | Given |
| 4. | q | MP: 1, 2 |
| 5. | r | MP: 3, 4 |

## Proofs can use equivalences too

Show that ¬p follows from p → q and ¬q

| | | |
|---|---|---|
| 1. | p → q | Given |
| 2. | ¬q | Given |
| 3. | ¬q → ¬p | Contrapositive: 1 |
| 4. | ¬p | MP: 2, 3 |

## Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.
- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. p → q given

2. (p ∨ r) → q ~~intro ∨ from 1.~~

Does not follow! e.g . p=F, q=F, r=T