CSE 311: Foundations of Computing I

Section 5: Number Theory & Induction Solutions

0. More Number Theory

(a) Prove that if $n^2 + 1$ is a perfect square, where n is an integer, then n is even.

Solution:

Suppose $n^2 + 1$ is a perfect square. Then, by definition of perfect square, $n^2 + 1 = k^2$ for some $k \in \mathbb{N}$. Since n and k are integers, we can define some integer z such that $k = n + z$. Now, substituting, we get:

$$\begin{aligned}
n^2 + 1 &= (n + z)^2 \\
n^2 + 1 &= n^2 + 2nz + z^2 \\
1 &= 2nz + z^2 \\
1 &= z(2n + z) \\
\frac{1}{z} &= (2n + z)
\end{aligned}$$

Since n and z are integers, $2n + z$ is an integer, which means $\frac{1}{z}$ is an integer. The only integers which satisfy this constraint are $z = \pm 1$, and in both these cases $z = \frac{1}{z}$, so we can subtract z from both sides to find $n = 0$ as the only solution. Since $n = 0$, and 0 is even, n is even.

(b) Prove that if n is a positive integer such that the sum of the divisors of n is $n + 1$, then n is prime.

Solution:

Note that $n \mid n$. If the sum of divisors of n is $n + 1$, then $n + 1 - n = 1$ must be the only other divisor. It follows, by definition of prime, that n is prime.

1. Induction

(a) Prove for all $n \in \mathbb{N}$ that if you have two groups of numbers, $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$, such that $\forall (i \in [n]).\ a_i \leq b_i$, then it must be that:

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i$$

Solution:

Let $\mathsf{P}(n)$ be the statement: "For any two groups of numbers, $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$, such that $\forall (i \in [n]).\ a_i \leq b_i$, it is true that:

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i \text{“}$$

defined for all $n \in \mathbb{N}$. We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n:

Base Case ($n = 0$). We know that:

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \sum_{i=1}^{0} a_i \\ &= 0 \\ &\leq 0 \\ &= \sum_{i=1}^{0} b_i \\ &= \sum_{i=1}^{n} b_i\end{aligned}$$

So the claim is true for $n = 0$.

Induction Hypothesis. Suppose that $\mathsf{P}(k)$ is true for some $k \in \mathbb{N}$.

Induction Step. Let the groups of numbers $a_1, \cdots, a_{k+1}$ and $b_1, \cdots, b_{k+1}$ be two groups such that $a_i \leq b_i$ for all $i \in [k+1]$.

Note that

$$\begin{aligned}\sum_{i=1}^{k+1} a_i &= \sum_{i=1}^{k} a_i + a_{k+1} && \text{[Splitting the summation]} \\ &\leq \sum_{i=1}^{k} b_i + a_{k+1} && \text{[By IH]} \\ &\leq \sum_{i=1}^{k} b_i + b_{k+1} && \text{[By Assumption]} \\ &\leq \sum_{i=1}^{k+1} b_i && \text{[Algebra]}\end{aligned}$$

Thus we have shown that if the claim is true for k, it is true for $k + 1$.

Therefore, we have shown $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction.

(b) For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = \sum_{i=1}^{n} i^2.$$

For all $n \in \mathbb{N}$, prove that $S_n = \frac{1}{6}n(n+1)(2n+1)$.

Solution:

Let $\mathsf{P}(n)$ be the statement "$S_n = \frac{1}{6}n(n+1)(2n+1)$" defined for all $n \in \mathbb{N}$. We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case. When $n = 0$, we know the sum of the squares of the first n positive integers is the sum of no terms, so we have a sum of 0. Thus, $S_0 = 0$. Since $\frac{1}{6}(0)(0+1)((2)(0)+1) = 0$, we know that $\mathsf{P}(0)$ is true.

Induction Hypothesis. Suppose that $\mathsf{P}(k)$ is true for some $k \in \mathbb{N}$.

Induction Step. Examining S_{k+1}, we see that

$$S_{k+1} = \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = S_k + (k+1)^2.$$

By the induction hypothesis, we know that $S_k = \frac{1}{6}k(k+1)(2k+1)$. Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned} S_{k+1} &= S_k + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= (k+1)\left(\frac{1}{6}k(2k+1) + (k+1)\right) \\ &= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\ &= \frac{1}{6}(k+1)\left(2k^2 + 7k + 6\right) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \end{aligned}$$

Thus, we can conclude that $\mathsf{P}(k+1)$ is true.

Therefore, because the base case and induction step hold, $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction.

(c) Define the triangle numbers as $\triangle_n = 1+2+\cdots+n$, where $n \in \mathbb{N}$. We showed in lecture that $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$\sum_{i=0}^{n} i^3 = \triangle_n^2$$

Solution:

First, note that $\triangle_n = \sum_{i=0}^{n} i$. So, we are trying to prove $\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$.

Let $\mathsf{P}(n)$ be the statement:

$$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$$

We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case. $0^3 = 0^2$, so $\mathsf{P}(0)$ holds.

Induction Hypothesis. Suppose that $\mathsf{P}(k)$ is true for some $k \in \mathbb{N}$.

Induction Step. We show P$(k+1)$:

$$
\begin{aligned}
\sum_{i=0}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 && \text{[Take out a term]} \\
&= \left(\sum_{i=0}^{k} i\right)^2 + (k+1)^3 && \text{[Induction Hypothesis]} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{[Substitution from part (a)]} \\
&= (k+1)^2 \left(\frac{k^2}{2^2} + (k+1)\right) && \text{[Factor } (k+1)^2\text{]} \\
&= (k+1)^2 \left(\frac{k^2+4k+4}{4}\right) && \text{[Add via comon denominator]} \\
&= (k+1)^2 \left(\frac{(k+2)^2}{4}\right) && \text{[Factor numerator]} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 && \text{[Take out the square]} \\
&= \left(\sum_{i=0}^{k+1} i\right)^2 && \text{[Substitution from part (a)]}
\end{aligned}
$$

Therefore, P(n) is true for all $n \in \mathbb{N}$ by induction.