CSE 311: Foundations of Computing I

Number Theory Solutions

These Three Threes

Prove that for all $n \in \mathbb{N}$ that $\sum_{i=0}^{n} 3^i = \frac{1}{2}(3^{n+1} - 1)$.

Solution:

Let $\mathsf{P}(n)$ be the statement $\sum_{i=0}^{n} 3^i = \frac{1}{2}(3^{n+1} - 1)$

We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case. $\sum_{i=0}^{0} 3^0 = 3^0 = 1 = \frac{1}{2}(2) = \frac{1}{2}(3^1 - 1)$, so $\mathsf{P}(0)$ holds.

Induction Hypothesis. Suppose that $\mathsf{P}(k)$ is true for some $k \in \mathbb{N}$.

Induction Step. We show $\mathsf{P}(k+1)$:

$$
\begin{aligned}
\sum_{i=0}^{k+1} 3^i &= \sum_{i=1}^{k} 3^i + 3^{k+1} && \text{[Take out a term]} \\
&= \frac{1}{2}(3^{k+1} - 1) + 3^{k+1} && \text{[Induction Hypothesis]} \\
&= \frac{1}{2}3^{k+1} - \frac{1}{2} + 3^{k+1} && \text{[Distributivity]} \\
&= \frac{3}{2}3^{k+1} - \frac{1}{2} && \text{[Addition]} \\
&= \frac{1}{2}3^{k+1+1} - \frac{1}{2} && \text{[Algebraic Properties of Real Numbers]} \\
&= \frac{1}{2}(3^{k+1+1} - 1) && \text{[Distributivity]}
\end{aligned}
$$

Therefore, $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction.