CSE 311: Foundations of Computing I

Section 4: Sets and Modular Arithmetic

0. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say so.

(a) $A = \{1, 2, 3, 2\}$

(b) $B = \{\{\}, \{\{\}\}, \{\{\}, \{\}\}, \{\{\}, \{\}, \{\}\}, \dots\}$

(c) $C = A \times (B \cup \{7\})$

(d) $D = \varnothing$

(e) $E = \{\varnothing\}$

(f) $F = \mathcal{P}(\{\varnothing\})$

1. Set = Set

Prove the following set identities.

(a) Let the universal set be $\mathcal{U}$. Prove $\overline{\overline{X}} = X$ for any set X.

(b) Prove $(A \oplus B) \oplus B = A$ for any sets A, B.

(c) Prove $A \cup B \subseteq A \cup B \cup C$ for any sets A, B, C.

(d) Let the universal set be $\mathcal{U}$. Prove $A \cap \overline{B} \subseteq A \setminus B$ for any sets A, B.

2. Casting Out Nines

Let $n \in \mathbb{N}$. Prove that if $n \equiv 0 \pmod 9$, then the sum of the digits of n is a multiple of 9.
You may use without proof that $a \equiv b \pmod m \to a^i \equiv b^i \pmod m$ for $i \in \mathbb{N}$.

3. Modular Arithmetic

(a) Prove that if $a \mid b$ and $b \mid a$, where a and b are integers, then $a = b$ or $a = -b$.

(b) Prove that if $n \mid m$, where n and m are integers greater than 1, and if $a \equiv b \pmod m$, where a and b are integers, then $a \equiv b \pmod n$.

4. New Definitions

- We say $(\mathcal{M}, \star)$ is a *magma* iff $\forall (x \in \mathcal{M}) \forall (y \in \mathcal{M})\ x \star y \in \mathcal{M}$.
- We say "e is a *left-identity*, in a magma $(\mathcal{M}, \star)$, iff $\forall (a \in \mathcal{M})\ e \star a = a$.
- We say "e is a *right-identity*, in a magma $(\mathcal{M}, \star)$, iff $\forall (a \in \mathcal{M})\ a \star e = a$.
- We say "x^{-1} is a *right-inverse of* x, in a magma $(\mathcal{M}, \star)$, iff for all right-identities, e, in $\mathcal{M}$, $x \star x^{-1} = e$.

(a) Let $(\mathcal{Q}, \triangle)$ be a magma. Prove that if a and b are both right-identities and all $m \in \mathcal{Q}$ have right-inverses, then $a = b$.

(b) Let $(\mathcal{R}, \square)$ be an associative magma with a left and right identity $e \in \mathcal{R}$. Prove for all $a \in \mathcal{R}$, if a has a right-inverse a^{-1}, then $(a^{-1})^{-1} = a$.