CSE 311: Foundations of Computing I

Section 10: Final Exam Review

0. Induction

Let f_n and g_n be defined as:

$$
\begin{aligned}
f_0 &= 0 \\
f_1 &= 1 \\
f_2 &= 1 \\
f_n &= f(n-1) + f(n-2) + f(n-3)
\end{aligned}
$$

$$
\begin{aligned}
g_0 &= 0 \\
g_1 &= 1 \\
g_n &= g(n-1) + g(n-2)
\end{aligned}
$$

Prove that $f_n \leq 3^n$ for all $n \geq 0$ and $f_n \geq g_n$ for all $n \geq 2$ by strong induction.

1. Structural Induction

Consider some new programs on lists:

$$
\begin{aligned}
&\mathsf{in}(a, [\,]) && = \mathsf{false} \\
&\mathsf{in}(a, b :: S) && = \mathsf{if}\ a = b\ \mathsf{then\ true\ else\ in}(a, S) \\
\\
&\mathsf{set}([\,]) && = [\,] \\
&\mathsf{set}(a :: S) && = \mathsf{not\ in}(a, S)\ \mathsf{and\ set}(S) \\
\\
&\mathsf{append}(a, [\,]) && = a :: [\,] \\
&\mathsf{append}(a, b :: L) && = b :: \mathsf{append}(a, L) \\
\\
&\mathsf{rev}([\,]) && = [\,] \\
&\mathsf{rev}(a :: L) && = \mathsf{append}(a, \mathsf{rev}(L)) \\
\\
&\mathsf{getAll}(a, [\,]) && = [\,] \\
&\mathsf{getAll}(a, b :: L) && = \mathsf{if}\ a = b\ \mathsf{then}\ b :: \mathsf{getAll}(a, L)\ \mathsf{else\ getAll}(a, L)
\end{aligned}
$$

Suppose that for arbitrary p, q, and list L, $\mathsf{getAll}(p, q :: L)) = \mathsf{getAll}(p, \mathsf{append}(q, L))$. Prove that if $\mathsf{set}(L)$, then $\mathsf{getAll}(a, L) = \mathsf{getAll}(a, \mathsf{rev}(L))$ for all lists L and elements a.

2. Regular Expressions, CFGs, and FSMs

Let $\Sigma = \{\mathtt{H}, \mathtt{J}, \mathtt{K}, \mathtt{L}\}$. Let the language L be defined for Σ^* such that $w \in L$ iff w:

- starts with `K` and ends with `L` or starts with `L` and ends with `K`
- has exactly one `J` between any two (not necessarily consecutive) occurrences of `K`
- has exactly one `H` between any two (not necessarily consecutive) occurrences of `L`

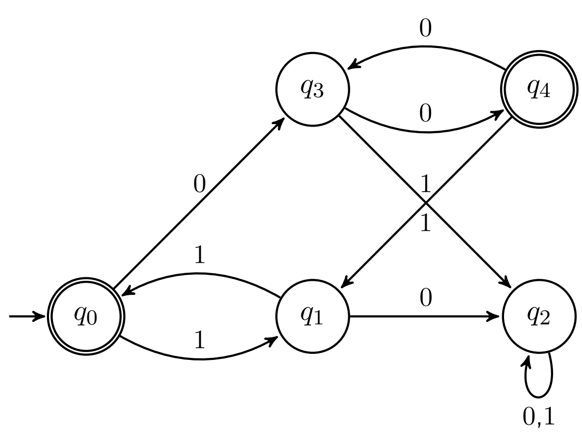
(a) Write a regular expression that matches L.

(b) Construct a CFG that generates L.

(c) Construct an NFA that accepts L.

3. DFA Minimization

Minimize the following DFA:



4. Irregularity

Let $\Sigma = \{A, C, G, T\}$. Let $\overline{w}$ be defined for a string $w \in \Sigma*$ such that for each character w_i, the character $\overline{w_i}$ is the complement of w_i. C and G are complements of each other, as are A and T. Prove that $\{w\overline{w} \in \Sigma^*\}$ is not regular.

5. Diagonalization

Here is a "proof" that the positive rationals are uncountable.

> Suppose for contradiction that the positive rationals $\mathbb{Q}^+$ are countable. Then there exists some listing of all elements $\mathbb{Q}^+ = \{q_1, q_2, q_3, \dots\}$. Note that each of these rationals q_i can also be written as an infinite decimal expansion. We define a new number $X \in \mathbb{Q}^+$ by flipping the diagonals of $\mathbb{Q}^+$; we set the ith digit of X to 7 if the ith digit of q_i is a 4, otherwise we set the digit to 4. This means that X differs from every q_i on the ith digit, so X *cannot* be one of q_i. Therefore our listing for $\mathbb{Q}^+$ was incomplete, which is a contradiction. Since the above proof works for any listing of the positive rationals $\mathbb{Q}^+$, *no* listing can be created for $\mathbb{Q}^+$, and therefore $\mathbb{Q}^+$ is uncountable.

What is the key error in this proof?

6. Cardinality

(a) You are a pirate. You begin in a square on a 2D grid which is infinite in all directions. In other words, wherever you are, you may move up, down, left, or right. Some single square on the infinite grid has treasure on it. Find a way to ensure you find the treasure in finitely many moves.

(b) Prove that $\{3x \ : \ x \in \mathbb{N}\}$ is countable.

(c) Prove that the set of irrational numbers is uncountable.
Hint: Use the fact that the rationals are countable and that the reals are uncountable.

(d) Prove that $\mathcal{P}(\mathbb{N})$ is uncountable.

7. Relations

Recall the following definitions of a relation R on A:

R is reflexive iff $(a, a) \in R$ for every $a \in A$.

R is symmetric iff $(a, b) \in R$ implies $(b, a) \in R$.

R is anti-symmetric iff $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$.

R is transitive iff $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

(a) Suppose that R is reflexive. Prove that $R \subseteq R^2$.

(b) Consider the relation $R = \{(x, y) \ : \ x = y + 1\}$ on $\mathbb{N}$. Is R reflexive? Transitive? Symmetric? Anti-symmetric?

R is transitive iff $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

(c) Suppose that R is reflexive. Prove that $R \subseteq R^2$.

(d) Consider the relation $R = \{(x, y) \ : \ x = y + 1\}$ on $\mathbb{N}$. Is R reflexive? Transitive? Symmetric? Anti-symmetric?

(e) Consider the relation $S = \{(x, y) \ : \ x^2 = y^2\}$ on $\mathbb{R}$. Prove that S is reflexive, transitive, and symmetric.

8. Uncomputability

(a) Let $\Sigma = \{0, 1\}$. Prove that the set of palindromes is decidable.

(b) Prove that the set $\{(\texttt{CODE}(P), x, y) \ : \ P$ is a program and $P(x) \neq P(y)\}$ is undecidable.