CSE 311: Foundations of Computing I

Section : More Midterm Review Solutions

0. Dividing by Nines

Prove that $9 \mid n^3 + (n+1)^3 + (n+2)^3$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

Induction Hypothesis: Assume that $9 \mid j^3 + (j+1)^3 + (j+2)^3$ for some arbitrary integer $j > 1$. Note that this is equivalent to assuming that $j^3 + (j+1)^3 + (j+2)^3 = 9k$ for some integer k.

Induction Step: $\boxed{\text{Goal: Show } 9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3}$

Now

$$\begin{aligned}
(j+1)^3 + (j+2)^3 + (j+3)^3 &= (j+3)^3 + 9k - j^3 \text{ for some integer } k \qquad \text{[Induction Hypothesis]} \\
&= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\
&= 9j^2 + 27j + 27 + 9k \\
&= 9(j^2 + 3j + 3 + k)
\end{aligned}$$

So $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$, so $P(j) \to P(j+1)$ for some arbitrary integer $j > 1$.

Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

1. Those 2's Just Grow Up So Fast

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Solution:

Let $P(n)$ be "$6n + 6 < 2^n$". We will prove $P(n)$ for all integers $n \geq 6$ by induction.

Base Case $(n = 6)$**:** $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

Induction Hypothesis: Assume that $6j + 6 < 2^j$ for an arbitrary integer $j \geq 6$.

Induction Step: $\boxed{\text{Goal: Show } 6(j+1) + 6 < 2^{j+1}}$

Now

$$\begin{aligned}
6(j+1) + 6 &= 6j + 6 + 6 \\
&< 2^j + 6 && \text{[Induction Hypothesis]} \\
&< 2^j + 2^j && \text{[Since } 2^j > 6 \text{, since } j \geq 6 \text{]} \\
&< 2 \cdot 2^j \\
&= 2^{j+1}
\end{aligned}$$

So $P(j) \to P(j+1)$ for an arbitrary integer $j \geq 6$.

Conclusion: $P(n)$ holds for all integers $n \geq 6$ by induction.

2. Proof by Harmonicas

Define

$$H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for $n \in \mathbb{N}$.

Solution:

We define H_i more formally as $\sum_{k=1}^{i} \frac{1}{k}$. Let $P(n)$ be "$H_{2^n} \geq 1 + \frac{n}{2}$". We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

Base Case $(n = 0)$**:** $H_{2^0} = H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq 1 + \frac{0}{2}$, so $P(0)$ holds.

Induction Hypothesis: Assume that $H_{2^j} \geq 1 + \frac{j}{2}$ for some arbitrary integer $j \in \mathbb{N}$.

Induction Step: $\boxed{\text{Goal: Show } H_{2^{j+1}} \geq 1 + \frac{j+1}{2}}$

Now

$$\begin{aligned}
H_{2^{j+1}} &= \sum_{k=1}^{2^{j+1}} \frac{1}{k} \\
&= \sum_{k=1}^{2^j} \frac{1}{k} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \\
&\geq 1 + \frac{j}{2} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} && \text{[Induction Hypothesis]} \\
&\geq 1 + \frac{j}{2} + 2^j \cdot \frac{1}{2^{j+1}} && \text{[There are } 2^j \text{ terms in } [2^j+1, 2^{j+1}] \text{ and each is at least } \frac{1}{2^{j+1}}\text{]} \\
&\geq 1 + \frac{j}{2} + \frac{2^j}{2^{j+1}} \\
&\geq 1 + \frac{j}{2} + \frac{1}{2} \\
&\geq 1 + \frac{j+1}{2}
\end{aligned}$$

So $P(j) \to P(j+1)$ for an arbitrary integer $j \in \mathbb{N}$.

Conclusion: $P(n)$ holds for all integers $n \in \mathbb{N}$ by induction.

3. Odds and Ends

Prove that for any even integer, there exists an odd integer greater than that even integer.

Solution:

Let x be an arbitrary even integer. By the definition of even, we know $x = 2y$ for some corresponding integer y. Now, we define z to be the integer $2y + 1$, which is odd by the definition of odd. By the basic properties of inequalities, we know $2y + 1 > 2y$ regardless of y, so we also know $z > x$.

We've now shown that there exists some integer z which is both odd and greater than x. Since x was arbitrary, we can generalize our conclusion to all even integers.

4. Magic Squares

Prove that if a real number $x \neq 0$, then $x^2 + \frac{1}{x^2} \geq 2$.

Solution:

Note that $(x^2-1)^2 \geq 0$, because all squares are at least 0. Distributing, we see that $x^4 + 1 \geq 2x^2$. Since $x \neq 0$, we can divide by x^2 to get $x^2 + \frac{1}{x^2} \geq 2$, which is what we were trying to prove.

Note: The first step may seem like "magic", but the way we generally solve these sorts of problems is by working backward and reversing the entire proof. You are allowed to introduce new facts while doing so, as long as it's clear why the fact is always true.

5. Primality Checking

When brute forcing if the number p is prime, you only need to check possible factors up to $\sqrt{p}$. In this problem, you'll prove why that is the case. Prove that if $n = ab$, then either a or b is at most $\sqrt{n}$.

Solution:

Suppose that $n = ab$, where n, a, and b are arbitrary integers. Now suppose without loss of generality that $a > \sqrt{n}$. Note that $n = ab > \sqrt{n}b$. Then, dividing both sides by $\sqrt{n}$, we get $\sqrt{n} > b$.

6. Even More Negative

Show that $\forall(x \in \mathbb{Z}).\ (\text{Even}(x) \rightarrow (-1)^x = 1)$

Solution:

Let $x \in \mathbb{Z}$ be arbitrary, and suppose x is even. Choose an $n \in \mathbb{Z}$ such that $x = 2n$ (we know such an n exists by the definition of even). It follows that we can rewrite this as

$$\begin{aligned}
(-1)^x &= (-1)^{2n} && [x = 2n] \\
&= ((-1)^2)^n \\
&= (1)^n && [(-1)^2 = 1] \\
&= 1 && [\forall n \in \mathbb{R},\ 1^n = 1]
\end{aligned}$$

We let x be arbitrary, assumed that it was even, and then showed that the implication holds. it follows that for every even integer x, $(-1)^x = 1$. Thus the claim is proven.

7. That's Odd...

Prove that every odd natural number can be expressed as the difference between two consecutive perfect squares.

Solution:

Let $x \in \mathbb{N}$ be arbitrary, and suppose it is odd. Using the definition of odd, choose a $k \in \mathbb{N}$ such that $x = 2k+1$. Now define n as $k+1$ (which is also a natural number, since $\mathbb{N}$ is closed under addition), such that $k = n-1$. Thus, $x = 2k+1 = 2(n-1)+1 = 2n-1$. Starting with this, we have

$$\begin{aligned}
x = 2n - 1 &= n^2 - n^2 + 2n - 1 && [\text{Add and subtract } n^2] \\
&= n^2 - (n^2 - 2n + 1) && [\text{rewrite}] \\
&= n^2 - (n-1)^2 && [\text{factoring}]
\end{aligned}$$

Thus, x can be expressed as the difference between n^2 and $(n-1)^2$, both of which are, by definition, perfect squares. Since x was arbitrary, it follows that any odd integer can be expressed as the difference between two squares.

8. United We Stand

We say that a set S is closed under an operation $\star$ iff $\forall(x, y \in S).\ (x \star y \in S)$.

(a) Prove $\mathbb{Z}$ is closed under $-$.

Solution:

Let $a, b \in \mathbb{Z}$ be arbitrary. We want to show that $(a - b) \in \mathbb{Z}$. By the theorem given, since $b \in \mathbb{Z}$, we know that $-b \in \mathbb{Z}$. Thus, we can rewrite $a - b$ as $a + (-b)$, which is addition between two integers. We are given that $\mathbb{Z}$ is closed under addition, so this number is also an integer. It follows that $\mathbb{Z}$ is closed under subtraction.

(b) Prove that $\mathbb{Z}$ is *not* closed under $/$.

Solution:

Choose $a = 1$ and $b = 2$. Both of these are integers. $\frac{a}{b} = \frac{1}{2} = 0.5$ which is not an integer. Thus, we have provided a counter-example to prove that the integers are *not* closed under division.

(c) Prove that $\mathbb{I}$ is *not* closed under $+$.

Solution:

To prove that $\mathbb{I}$ is not closed under addition, we need to prove the negation of the definition of closure. That is, we want to show $\exists(x, y \in \mathbb{I}).\ (x + y \notin \mathbb{I})$.

Choose $x = \sqrt{2}$ and choose $y = -\sqrt{2}$. Note that both of these are irrational (we'll prove this later). Also, note that $x + y = \sqrt{2} - \sqrt{2} = 0$. We can write 0 as $\frac{0}{1}$. Since 0 and 1 are both integers, this shows that $x + y$ is a rational number by definition of rational numbers.

This means that $x+y$ is not irrational, proving the claim that the irrationals are not closed under addition.

9. A Hint of Things to Come

Prove that $\forall(a, b \in \mathbb{Z}).\ a^2 - 4b \neq 2$.

Solution:

Let a, b be arbitrary integers. We go by cases on a.

Case: a is even: By definition of even, there exists an integer k such that $a = 2k$. Substituting, we have $(2k)^2 - 4b = 4k^2 - 4b = 4(k^2 - 4b)$. $k^2 - 4b$ is an integer because integers are closed under addition and multiplication, so this is an integer multiple of 4. Note that 2 is not an integer multiple of 4, so $4(k^2 - 4b)$ cannot equal 2.

Case: a is odd: By definition of odd, there exists an integer j such that $a = 2j + 1$. Substituting, we have $(2j + 1)^2 - 4b = 4j^2 + 4j + 1 - 4b = 4(j^2 + j - b) + 1$. Note that $j^2 + j - b$ is an integer because integers are closed under addition and multiplication, so $4(j^2 + j - b) + 1$ is odd by the definition of odd. $2 = 2 * 1$, so 2 is even by the definition of even, and since an odd integer cannot equal an even integer, $4(j^2 + j - b) + 1$ cannot equal 2.

Since we have shown for all cases of integer a and arbitrary integer b, $a^2 - 4b \neq 2$, the claim is proven.

10. Proofs or it didn't happen!

(a) Prove that if x is an odd integer and y is an integer, then xy is odd if and only if y is odd.

Solution:

Let x be an arbitrary odd integer and y be an arbitrary integer. By the definition of odd, we know $x = 2k + 1$ for some corresponding integer k. We will show that if y is odd, xy is odd, and if y is even, xy is even to show both directions of the biconditional.

Suppose y is odd. Then by the definition of odd, we know $y = 2m + 1$ for some corresponding integer m. Then, $xy = (2k + 1)(2m + 1) = 4km + 2m + 2k + 1 = 2(2km + m + k) + 1$. Since $2km + m + k$ is an integer because m and k are integers, xy is odd by the definition of odd.

Suppose y is even. Then by the definition of even, we know $y = 2n$ for some corresponding integer n. Then, $xy = (2k + 1)(2n) = 2(2kn + n)$. Since $2kn + n$ is an integer because n and k are integers, xy is even by the definition of even, so xy is odd if and only if y is odd. Since x and y were arbitrary, we can generalize our conclusion to all odd integers x and all integers y.

(b) Prove that for integers x and y, if $(x + y)^2 = 16$ that $xy < 10$.

Solution:

Let x and y be arbitrary integers such that $(x + y)^2 = 16$.

First, notice that $(x + y)^2 = x^2 + 2xy + y^2 = 16$ by distributivity. Now, subtracting $x^2 + y^2$ from both sides of our equation, we see that $2xy = 16 - x^2 - y^2$.

But because squares are positive, we know that $x^2 \geq 0$ and $y^2 \geq 0$. If we respectively subtract x^2 and y^2 from both sides of these inequalities, we obtain $0 \geq -x^2$ and $0 \geq -y^2$. Now, we see:

$$\begin{aligned} 0 + 0 &\geq -x^2 - y^2 && \text{[Addition of inequalities]} \\ 16 &\geq 16 - x^2 - y^2 && \text{[Adding 16 to both sides]} \end{aligned}$$

Composing this inequality with our initial identitiy, we have $2xy = 16 - x^2 - y^2 \leq 16$, or equivalently, $2xy \leq 16$. Dividing both sides by 2, we see that $xy \leq 8$. Finally, since $8 < 10$, it must be the case that $xy \leq 8 < 10$, so $xy < 10$. This is what we wanted to show.

(c) Prove that for positive integers x, a where x is odd, there is an even integer y such that $a^x \leq a^y$.

Solution:

Let x be an arbitrary odd integer and a be an arbitrary positive integer. By the definition of odd, we know $x = 2k + 1$ for some corresponding integer k. Then we have

$$\begin{aligned} a^x &= a^{2k+1} && \text{[Substituting for } x\text{]} \\ &\leq a \cdot a^{2k+1} && \text{[}a\text{ is a positive integer, so } a \geq 1\text{]} \\ &= a^{2k+2} && \text{[Properties of exponents]} \end{aligned}$$

We have $2k + 2 > 2k + 1$, and $2k + 2 = 2(k + 1)$, which is even since $k + 1$ is an integer. Thus, we have found an even integer y such that $a^x \leq a^y$, so we have shown that our claim holds.

11. To B or not to B

Prove $(A \setminus B) \cap B = \varnothing$.

Solution:

$$\begin{aligned}
(A \setminus B) \cap B &= \{x \ : \ x \in A \setminus B \wedge x \in B\} && \text{[Definition of } \cap\text{]} \\
&= \{x \ : \ (x \in A \wedge x \notin B \wedge x \in B)\} && \text{[Definition of } \setminus\text{]} \\
&= \{x \ : \ (x \in A \wedge F)\} && \text{[Negation]} \\
&= \{x \ : \ F\} && \text{[Domination]} \\
&= \varnothing && \text{[Definition of } \varnothing\text{]}
\end{aligned}$$

It follows that $(A \setminus B) \cap B = \varnothing$.