CSE 311: Foundations of Computing I

Section 5: Midterm Practice

0. Propositional Logic

(a) Is the following expression a contingency, contradiction, or tautology?

$$(p \to q) \land (q \to r) \to (p \to r)$$

(b) Show that $\neg p \to (q \to r)$ and $q \to (p \lor r)$ are logically equivalent.

1. Predicate Logic

Let the domain of discourse be all plants and leaves. You may use the predicates HasLeaf(x, y) ::= "x has y as a leaf", Equals(x, y) ::= "x is the same object as y", Leaf(x) ::= "x is a leaf" and Plant(x) ::= "x is a plant", IsPurple(x) be "x is purple", IsGolden(x) be "x is golden", and let the constant LuckyLeaf be the Lucky Leaf.

Translate the following sentences to predicate logic using quantifiers.

(a) Every plant has at least 2 leaves.

(b) Every plant has at most 2 leaves.

(c) There is exactly one plant that has no leaves.

(d) If a plant has the Lucky Leaf, all other leaves on that plant are golden, but the Lucky Leaf is purple, and then no other plants have golden or purple leaves.

2. Proofs with Number Theory

(a) Prove that if $n^2 + 1$ is a perfect square, where n is an integer, then n is even.

(b) Prove that if n is a positive integer such that the sum of the divisors of n is $n + 1$, then n is prime.

3. Induction

(a) Prove for all $n \in \mathbb{N}$ that if you have two groups of numbers, $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$, such that $\forall(i \in [n]).\ a_i \leq b_i$, then it must be that:

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i$$

(b) For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = \sum_{i=1}^{n} i^2.$$

For all $n \in \mathbb{N}$, prove that $S_n = \frac{1}{6}n(n+1)(2n+1)$.

(c) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. Theorem: $\triangle_n = \frac{n(n+1)}{2}$.

Prove the following equality for all $n \in \mathbb{N}$:

$$\sum_{i=0}^{n} i^3 = \triangle_n^2$$

(d) Prove that $9 \mid n^3 + (n+1)^3 + (n+2)^3$ for all $n > 1$ by induction.

(e) Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

(f) Define

$$H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for $n \in \mathbb{N}$.

4. Set Proofs

Prove for any sets A and B, $P((A \cup B) \setminus B) \subseteq P(A)$.