CSE 311: Foundations of Computing I

Section 5: Midterm Practice Solutions

0. Propositional Logic

(a) Is the following expression a contingency, contradiction, or tautology?

$$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$$

Solution:

Tautology.

(b) Show that $\neg p \rightarrow (q \rightarrow r)$ and $q \rightarrow (p \vee r)$ are logically equivalent.

Solution:

Solution 1: Truth Table

p	q	r	$\neg p$	$q \rightarrow r$	$\neg p \rightarrow (p \rightarrow r)$
T	T	T	F	T	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

p	q	r	$p \vee r$	$q \rightarrow (p \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	T

Since the last column is identical on both tables the two statements must be equivalent.

Solution 2: Formal Proof

$$\begin{aligned}
\neg p \rightarrow (q \rightarrow r) &= \neg\neg p \vee (q \rightarrow r) && \text{By Law of Implication} \\
&= p \vee (q \rightarrow r) && \text{By Double Negation} \\
&= p \vee (\neg q \vee r) && \text{By Law of Implication} \\
&= (\neg q \vee r) \vee p && \text{By Commutativity} \\
&= \neg q \vee (r \vee p) && \text{By Associativity} \\
&= \neg q \vee (p \vee r) && \text{By Commutativity} \\
&= q \rightarrow (p \vee r) && \text{By Law of Implication}
\end{aligned}$$

1. Predicate Logic

Let the domain of discourse be all plants and leaves. You may use the predicates HasLeaf(x, y) ::= "x has y as a leaf", Equals(x, y) ::= "x is the same object as y", Leaf(x) ::= "x is a leaf" and Plant(x) ::= "x is a plant", IsPurple(x) be "x is purple", IsGolden(x) be "x is golden", and let the constant LuckyLeaf be the Lucky Leaf.

Translate the following sentences to predicate logic using quantifiers.

(a) Every plant has at least 2 leaves.

Solution:

$$\forall x\ (\text{Plant}(x) \rightarrow \exists y \exists z\ (\neg\text{Equals}(y, z) \wedge \text{HasLeaf}(x, y) \wedge \text{HasLeaf}(x, z)))$$

(b) Every plant has at most 2 leaves.

Solution:

$$\forall x\ (\text{Plant}(x) \rightarrow$$
$$\forall a \forall b \forall c\ ((\text{HasLeaf}(x, a) \wedge \text{HasLeaf}(x, b) \wedge \text{HasLeaf}(x, c)) \rightarrow (\text{Equals}(a, b) \vee \text{Equals}(a, c) \vee \text{Equals}(b, c))))$$

(c) There is exactly one plant that has no leaves.

Solution:

$$\exists x\ (\text{Plant}(x) \wedge \neg(\exists y(\text{HasLeaf}(x, y))) \wedge \forall z\ ((\text{Plant}(z) \wedge \neg(\exists k\ (\text{HasLeaf}(z, k)))) \rightarrow \text{Equals}(x, z)))$$

(d) If a plant has the Lucky Leaf, all other leaves on that plant are golden, but the Lucky Leaf is purple, and then no other plants have golden or purple leaves.

Solution:

$$\text{IsPurple}(\text{LuckyLeaf}) \wedge \forall x\ (\text{Plant}(x) \wedge \text{HasLeaf}(x, \text{LuckyLeaf}) \rightarrow$$
$$\forall y\ ((\text{HasLeaf}(x, y) \wedge \neg\text{Equal}(y, \text{LuckyLeaf})) \rightarrow$$
$$\text{IsGolden}(y)) \wedge \forall z\ ((\text{Plant}(z) \wedge \neg\text{Equal}(x, z)) \rightarrow$$
$$\forall k\ (\text{HasLeaf}(z, k) \rightarrow (\neg\text{IsPurple}(k) \wedge \neg\text{IsGolden}(k)))))$$

2. Proofs with Number Theory

(a) Prove that if $n^2 + 1$ is a perfect square, where n is an integer, then n is even.

Solution:

Let n be an arbitrary integer, and suppose $n^2 + 1$ is a perfect square. Then, by definition of perfect square, $n^2 + 1 = k^2$ for some $k \in \mathbb{N}$. Since n and k are integers, we can define some integer z such that $k = n + z$. Now, substituting, we get:

$$\begin{aligned} n^2 + 1 &= (n+z)^2 \\ n^2 + 1 &= n^2 + 2nz + z^2 \\ 1 &= 2nz + z^2 \\ 1 &= z(2n+z) \\ \frac{1}{z} &= (2n+z) \end{aligned}$$

Since n and z are integers, $2n + z$ is an integer, which means $\frac{1}{z}$ is an integer. The only integers which satisfy this constraint are $z = \pm 1$, and in both these cases $z = \frac{1}{z}$, so we can subtract z from both sides to find $n = 0$ as the only solution. Since $n = 0$, and 0 is even, n is even.

(b) Prove that if n is a positive integer such that the sum of the divisors of n is $n + 1$, then n is prime.

Solution:

Let n be an arbitrary positive integer, and suppose the sum of the divisors of n is $n+1$. Note that $n \mid n$. Since the sum of divisors of n is $n + 1$, $n + 1 - n = 1$ must be the only other divisor. It follows, by definition of prime, that n is prime. Since we supposed n was a positive integer whose divisors sum to $n + 1$, we see that for any arbitrary positive integer n, if n's divisors sum to $n + 1$, n must be prime.

3. Induction

(a) Prove for all $n \in \mathbb{N}$ that if you have two groups of numbers, $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$, such that $\forall(i \in [n]).\ a_i \leq b_i$, then it must be that:

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i$$

Solution:

Let $\mathsf{P}(n)$ be the statement: "For any two groups of numbers, $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$, such that $\forall(i \in [n]).\ a_i \leq b_i$, it is true that:

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i''$$

defined for all $n \in \mathbb{N}$. We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n:

Base Case ($n = 0$). We know that:

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \sum_{i=1}^{0} a_i \\ &= 0 \\ &\leq 0 \\ &= \sum_{i=1}^{0} b_i \\ &= \sum_{i=1}^{n} b_i\end{aligned}$$

So the claim is true for $n = 0$.

Induction Hypothesis. Suppose that $\mathsf{P}(k)$ is true for some $k \in \mathbb{N}$.

Induction Step. Let the groups of numbers $a_1, \cdots, a_{k+1}$ and $b_1, \cdots, b_{k+1}$ be two groups such that $a_i \leq b_i$ for all $i \in [k+1]$.

Note that

$$\begin{aligned}\sum_{i=1}^{k+1} a_i &= \sum_{i=1}^{k} a_i + a_{k+1} && \text{[Splitting the summation]} \\ &\leq \sum_{i=1}^{k} b_i + a_{k+1} && \text{[By IH]} \\ &\leq \sum_{i=1}^{k} b_i + b_{k+1} && \text{[By Assumption]} \\ &\leq \sum_{i=1}^{k+1} b_i && \text{[Algebra]}\end{aligned}$$

Thus we have shown that if the claim is true for k, it is true for $k + 1$.

Therefore, we have shown $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction.

(b) For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = \sum_{i=1}^{n} i^2.$$

For all $n \in \mathbb{N}$, prove that $S_n = \frac{1}{6}n(n+1)(2n+1)$.

Solution:

Let $\mathsf{P}(n)$ be the statement "$S_n = \frac{1}{6}n(n+1)(2n+1)$" defined for all $n \in \mathbb{N}$. We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case. When $n = 0$, we know the sum of the squares of the first n positive integers is the sum of no terms, so we have a sum of 0. Thus, $S_0 = 0$. Since $\frac{1}{6}(0)(0+1)((2)(0)+1) = 0$, we know that $\mathsf{P}(0)$ is true.

Induction Hypothesis. Suppose that $\mathsf{P}(k)$ is true for some $k \in \mathbb{N}$.

Induction Step. Examining S_{k+1}, we see that

$$S_{k+1} = \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = S_k + (k+1)^2.$$

By the induction hypothesis, we know that $S_k = \frac{1}{6}k(k+1)(2k+1)$. Therefore, we can substitute and rewrite the expression as follows:

$$\begin{aligned} S_{k+1} &= S_k + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= (k+1)\left(\frac{1}{6}k(2k+1) + (k+1)\right) \\ &= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\ &= \frac{1}{6}(k+1)\left(2k^2 + 7k + 6\right) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1) \end{aligned}$$

Thus, we can conclude that $\mathsf{P}(k+1)$ is true.

Therefore, because the base case and induction step hold, $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction.

(c) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. Theorem: $\triangle_n = \frac{n(n+1)}{2}$.
Prove the following equality for all $n \in \mathbb{N}$:

$$\sum_{i=0}^{n} i^3 = \triangle_n^2$$

Solution:

First, note that $\triangle_n = \sum_{i=0}^{n} i$. So, we are trying to prove $\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$.
Let $\mathsf{P}(n)$ be the statement:

$$\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$$

We prove that $\mathsf{P}(n)$ is true for all $n \in \mathbb{N}$ by induction on n.

Base Case. When $n = 0$, we have $0^3 = 0^2$, so $\mathsf{P}(0)$ holds.

Induction Hypothesis. Suppose that $\mathsf{P}(k)$ is true for some $k \in \mathbb{N}$.

Induction Step. We show P$(k+1)$:

$$\begin{aligned}
\sum_{i=0}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 && \text{[Take out a term]} \\
&= \left(\sum_{i=0}^{k} i\right)^2 + (k+1)^3 && \text{[Induction Hypothesis]} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{[Substitution from part (a)]} \\
&= (k+1)^2\left(\frac{k^2}{2^2} + (k+1)\right) && \text{[Factor } (k+1)^2\text{]} \\
&= (k+1)^2\left(\frac{k^2+4k+4}{4}\right) && \text{[Add via comon denominator]} \\
&= (k+1)^2\left(\frac{(k+2)^2}{4}\right) && \text{[Factor numerator]} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 && \text{[Take out the square]} \\
&= \left(\sum_{i=0}^{k+1} i\right)^2 && \text{[Substitution from part (a)]}
\end{aligned}$$

Therefore, P(n) is true for all $n \in \mathbb{N}$ by induction.

(d) Prove that $9 \mid n^3 + (n+1)^3 + (n+2)^3$ for all $n > 1$ by induction.

Solution:

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

Induction Hypothesis: Suppose that $P(j)$ is true for some $j > 1$. Note that this is equivalent to assuming that $j^3 + (j+1)^3 + (j+2)^3 = 9k$ for some integer k.

Induction Step: We want to show that $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$. First, note that our induction hypothesis $P(j)$ is equivalent to assuming that $j^3 + (j+1)^3 + (j+2)^3 = 9k$ for some integer k. Now

$$\begin{aligned}
(j+1)^3 + (j+2)^3 + (j+3)^3 &= (j+3)^3 + 9k - j^3 \text{ for some integer } k && \text{[Induction Hypothesis]} \\
&= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\
&= 9j^2 + 27j + 27 + 9k \\
&= 9(j^2 + 3j + 3 + k)
\end{aligned}$$

So $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$, so $P(j) \to P(j+1)$ for some arbitrary integer $j > 1$. Thus, $P(n)$ holds for all naturals $n > 1$ by induction.

(e) Prove that $6n + 6 < 2^n$ for all $n \geq 6$.

Solution:

Let $P(n)$ be "$6n+6 < 2^n$". We will prove $P(n)$ for all integers $n \geq 6$ by induction.

Base Case $(n=6)$**:** $6n+6 = 6 \cdot 6 + 6 = 42 < 64 = 2^6 = 2^n$, so $P(6)$ holds.

Induction Hypothesis: Suppose that $P(j)$ is true for some $j \geq 6$.

Induction Step: We want to show that $6(j+1)+6 < 2^{j+1}$

We have that

$$\begin{aligned}
6(j+1)+6 &= 6j+6+6 \\
&< 2^j + 6 && \text{[Induction Hypothesis]} \\
&< 2^j + 2^j && \text{[Since } 2^j > 6\text{, since } j \geq 6\text{]} \\
&< 2 \cdot 2^j \\
&< 2^{j+1}
\end{aligned}$$

So $P(j) \to P(j+1)$ for an arbitrary integer $j \geq 6$. Therefore, $P(n)$ holds for all naturals $n \geq 6$ by induction.

(f) Define

$$H_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ for $n \in \mathbb{N}$.

Solution:

We define H_i more formally as $\sum_{k=1}^{i} \frac{1}{k}$. Let $P(n)$ be "$H_{2^n} \geq 1 + \frac{n}{2}$". We will prove $P(n)$ for all $n \in \mathbb{N}$ by induction.

Base Case $(n=0)$**:** $H_{2^n} = H_{2^0} = H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq 1 + \frac{0}{2} = 1 + \frac{n}{2}$, so $P(0)$ holds.

Induction Hypothesis: Suppose $P(j)$ is true for some $j \in \mathbb{N}$.

Induction Step: We want to show that $H_{2^{j+1}} \geq 1 + \frac{j+1}{2}$

Now

$$\begin{aligned}
H_{2^{j+1}} &= \sum_{k=1}^{2^{j+1}} \frac{1}{k} \\
&= \sum_{k=1}^{2^j} \frac{1}{k} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \\
&\geq 1 + \frac{j}{2} + \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} && \text{[Induction Hypothesis]} \\
&\geq 1 + \frac{j}{2} + 2^j \cdot \frac{1}{2^{j+1}} && \text{[There are } 2^j \text{ terms in } [2^j+1, 2^{j+1}] \text{ and each is at least } \tfrac{1}{2^{j+1}}\text{]} \\
&\geq 1 + \frac{j}{2} + \frac{2^j}{2^{j+1}} \\
&\geq 1 + \frac{j}{2} + \frac{1}{2} \\
&\geq 1 + \frac{j+1}{2}
\end{aligned}$$

So $P(j) \to P(j+1)$ for an arbitrary integer $j \in \mathbb{N}$. Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by induction.

4. Set Proofs

Prove for any sets A and B, $P((A \cup B) \setminus B) \subseteq P(A)$.

Solution:

Let A, B be arbitrary sets. Let X be an arbitrary element of $P((A \cup B) \setminus B)$. Note that $X \subseteq (A \cup B) \setminus B$ by the definition of powersets. Let $x \in X$ be arbitrary. Then, by the definition of subsets $x \in ((A \cup B) \setminus B)$. Note that, $x \in (A \cup B) \land x \notin B$ by the definition of set difference. Then, $((x \in A \lor x \in B) \land x \notin B)$ by the definition of set union. Then, it follows that $x \in A \land x \notin B$. Then, it follows that $x \in A$. Since x was arbitrary, by the definition of subsets $X \subseteq A$. Then by the definition of powersets $X \in P(A)$. Since X was arbitrary, by the definition of subsets $P((A \cup B) \setminus B) \subseteq P(A)$.