CSE 311: Foundations of Computing I

QuickCheck: Logical Translations Solutions

0. Fish are Food, not Friends

Consider the following English statement:

> "Every fish in the Pacific Ocean that is not the smallest fish in the Pacific Ocean eats a smaller fish in the Pacific Ocean."

Let the domain of discourse be all fish and the Pacific Ocean.

Let Fish(x) be "x is a fish", LivesIn(x, y) be "x lives in y", Smaller(x, y) be "x is smaller than y", Equals(x, y) be "x equals y", and Eats(x, y) be "x eats y". Let PacificOcean represent the Pacific Ocean.

Express this statement as a logical expression using the specified domain and predicates.

Solution:

Solution 1:

One solution is to translate the sentence in pieces. We first translate the idea that "x is the smallest fish in the Pacific Ocean", then insert that smaller proposition into the overall one:

$$
\begin{array}{l}
\forall x \ (\\
\quad (\\
\qquad \mathsf{Fish}(x) \\
\qquad \wedge \mathsf{LivesIn}(x, \mathsf{PacificOcean}) \\
\qquad \wedge \neg \forall y \ ((\mathsf{Fish}(y) \wedge \neg \mathsf{Equals}(x, y) \wedge \mathsf{LivesIn}(y, \mathsf{PacificOcean})) \rightarrow \mathsf{Smaller}(x, y)) \\
\quad) \\
\quad \rightarrow \exists z \ (\mathsf{Fish}(z) \wedge \mathsf{LivesIn}(z, \mathsf{PacificOcean}) \wedge \mathsf{Smaller}(z, x) \wedge \mathsf{Eats}(x, z)) \\
)
\end{array}
$$

Solution 2:

Another solution is to "invert" the sentence slightly: rather then expressing that x is the smallest fish, we state that there must exist some fish *smaller* then x (and at the same time, say we eat it).

Initially, ending with just this exists clause seems sufficient, but what happens if our domain of course happens to contain just the Pacific Ocean and one fish (living in the Pacific)? In that case, our exists is false, so we end up with the implication $\mathsf{T} \rightarrow \mathsf{F} \equiv F$. However, if we consider the English sentence, we see it ought to evaluate to true in this case (since the sentence doesn't care what the smallest fish does).

So, to fix this, we modify the final clause to specifically whitelist this edge case.

$$
\begin{array}{l}
\forall x \ (\\
\quad (\mathsf{Fish}(x) \wedge \mathsf{LivesIn}(x, \mathsf{PacificOcean})) \\
\quad \rightarrow (\\
\qquad (\exists y \ (\mathsf{Fish}(y) \wedge \mathsf{LivesIn}(y, \mathsf{PacificOcean}) \wedge \mathsf{Smaller}(y, x) \wedge \mathsf{Eats}(x, y))) \\
\qquad \oplus (\forall z \ ((\mathsf{Fish}(z) \wedge \mathsf{LivesIn}(z, \mathsf{PacificOcean})) \rightarrow (\mathsf{Smaller}(x, z) \oplus \mathsf{Equals}(x, z)))) \\
\quad) \\
)
\end{array}
$$

Solution 3:

This solution, unlike solution 2, decides not to combine the "there exists a smaller fish" and "we eat a smaller fish" clauses into one. By doing so, we can move the "there exists a smaller fish" clause before the implication, rather then having it exist after. This does result in some redundancy, but lets us avoid having to "whitelist" specific cases.

Interestingly, this solution ends up being nearly identical to solution 1, if we had pushed the negation through the $\forall$. (The only difference is that after pushing the negation through, solution 1 ends up with the clause $\neg\mathsf{Smaller}(x, y)$, but this solution uses the clause $\mathsf{Smaller}(y, x)$ – notice that the arguments are swapped.)

$$
\begin{aligned}
&\forall x \ (\\
&\quad (\\
&\qquad \mathsf{Fish}(x) \\
&\qquad \wedge \mathsf{LivesIn}(x, \mathsf{PacificOcean}) \\
&\qquad \wedge \exists y \ (\mathsf{Fish}(y) \wedge \mathsf{LivesIn}(y, \mathsf{PacificOcean}) \wedge \mathsf{Smaller}(y, x)) \\
&\quad) \\
&\quad \rightarrow \exists z \ (\mathsf{Fish}(z) \wedge \mathsf{LivesIn}(z, \mathsf{PacificOcean}) \wedge \mathsf{Smaller}(z, x) \wedge \mathsf{Eats}(x, z)) \\
&)
\end{aligned}
$$