CSE 311: Foundations of Computing I

The Proof Is In The Pudding

The Claim

Suppose we're given a complicated claim, like the following:

$$\forall(a \in \mathbb{R})\ \forall(e > 0)\ \exists(d > 0)\ \forall(x \in \mathbb{R})\ (|x - a| < d \to |2x^2 - 2a^2| < e)$$

This handout will help you answer the following questions related to proving the claim:

- How do I start?
- How do I structure my proof?
- What do I think about when trying to write the proof?
- How does my thought process when proving the claim differ from the proof itself?
- What are some important concerns I should think about when writing proofs?

The key idea we will introduce here is the difference between "scratch work" and "the proof". In particular, we will have two running columns as we write the proof:

- One column of "what we're thinking"/"work to actually write the proof"
- One column of a "pristine proof" of the claim.

It's very important to *upfront* understand what does and does not belong in a proof. In particular, you may spend a good while solving equations, finding variables, etc.–none of this work belongs in the proof! A proof does not tell your reader "here is how I solved the problem"; a proof tells your reader "here is why the claim is true".

As such, you may find yourself asking the question "how did they come up with that function/variable/idea" when reading other people's proofs. In fact, if the proof is well-written, this should happen a lot. To demonstrate this to you, we'll begin with a "pristine" proof of the above claim.

Try to circle parts of the proof that you feel "came out of nowhere" (but nonetheless, you should be able to follow the logic).

The "Pristine" Proof

Let $a \in \mathbb{R}$ be arbitrary. Let $e > 0$ be arbitrary. Choose $d = \min\left(0.99, \frac{e}{2|2a|+2}\right)$. Let $x \in \mathbb{R}$ be arbitrary. Suppose $|x - a| < d$.

Then, $|x - a| < 1$ and $|x - a| < \frac{e}{2|2a|+2}$. Multiplying both sides of the second equation, we have:

$$e > 2|x - a|(|2a| + 1) > 2|x - a|(|2a| + |x - a|) > 2|x - a||x + a| = 2|x^2 - a^2| = |2x^2 - 2a^2|$$

Notice that we replaced the 1 with $|x - a|$, because we were able to assume $|x - a| < 1$ above.

Commentary on the "Pristine" Proof

Be honest. You probably fumbled to follow the proof a bit. Why? Probably because a bunch of algebra justifications were omitted. For example, at some point, we used that $|2a| + |x - a| \geq |x + a|$ which probably makes sense, but wouldn't you have rathered it be explained in the proof itself?

Be even more honest. You probably have no idea where $d = \min\left(0.99, \frac{e}{2|2a|+2}\right)$ came from. The crazy thing: That's okay! All the statement you were trying to prove says is "there exists a d"–and the prover (me!) found one for you.

Unfortunately, what remains is how you might come up with this on your own. But before getting there, we've already hit a hairy point about proofs. Why were we allowed to make the d in terms of e? Could we have used x?

Commentary on Variables in Proofs

You may have wondered why we've been so careful about defining variables in proofs thus far. It turns out the question at the end of the last section is the reason. In particular, take a look at the first few lines of the proof: "Let $a \in \mathbb{R}$ be arbitrary. Let $e > 0$ be arbitrary. Choose $d = \min\left(0.99, \frac{e}{2|2a|+2}\right)$. Let $x \in \mathbb{R}$."

Slightly more informally, what this means is (*in order*):

- Introduce a variable "a" (think like `int a` in code).
- Introduce a variable "e"
- Define a variable "d" (think like `int d = blah blah;`) The important point about d is, just as in programming, we can only use the variables currently in scope. This means d can be written in terms of a and e, *but NOT* x. This should intuitively make sense, so far, we've basically written the following code:

```
real d(real a, real e) {
   return min(0.99, e/(2*abs(2a) + 2);
}
```

  If a coder tried to use x in this function definition, it would be obviously wrong, right?

- Now, finally, define x.

A Point About "Reversibility" and Implications

Your goal in a proof is very often to prove an implication (or a series of implications!). You've seen that the approach we take here is to assume the first part of the implication and prove the second part. In fact, that's what the "pristine proof" does. Take a look:

- $|x-a| < 1$ and $|x-a| < d \to |x-a| < \frac{e}{2|2a|+2}$
- $|x-a| < 1$ and $|x-a| < \frac{e}{2|2a|+2} \to e > 2|x-a|(|2a|+1)$
- $|x-a| < 1$ and $e > 2|x-a|(|2a|+1) \to e > 2|x-a|(|2a|+|x-a|)$
- $e > 2|x-a|(|2a|+|x-a|) \to e > 2|x-a||x+a|$
- $e > 2|x-a||x+a| \to e > 2|x^2-a^2|$
- $e > 2|x^2-a^2| \to e > |2x^2-2a^2|$

But, of course, this looks horrible, and, so, we condense it as above.

Notice that these implications only go one way! Nowhere in here, did we assert (or prove) that these implications go in the other direction!

It turns out that there's very good reason for that. Consider $a = -10, x = 10, e = 6$. Then, note that

$$6 = e > |2x^2 - 2a^2| = |2(10)^2 - 2(-10)^2| = |0| = 0$$

But,

$$d = \min\left(0.99, \frac{e}{2|2|+2}\right) = \min\left(0.99, \frac{6}{2|2|+2}\right) = 0.99$$

That is, $20 = |x-a| < d = 0.99$ which is not true.
To compact this a bit, we proved that $|x-a| < d \to e > |2x^2 - 2a^2|$, but *disproved* that $e > |2x^2 - 2a^2| \to |x-a| < d$. This justifies the idea that **the direction of the implications is significant in proofs**!

The Main Event

We've shown you the "pristine proof" and some *gotchas*. Now, let's discuss how to actually go about *writing* this proof. A reminder of the claim:

$$\forall(a \in \mathbb{R})\ \forall(e > 0)\ \exists(d > 0)\ \forall(x \in \mathbb{R})\ (|x-a| < d \to |2x^2 - 2a^2| < e)$$

Proof	Commentary & Scratch Work
Let $a \in \mathbb{R}$ be arbitrary. Let $e > 0$ be arbitrary.	*The statement starts with two* $\forall$*'s. So, we define variables for them. Remember: we could use any variable we want here as long as we're consistent.*
	Now, we get stuck, because we need to choose a value that will work... but we have no idea what we need. Time for some scratch work... Looking ahead, we need to choose a d *that makes the implication true.*
	We will work backwards. But we will be cautious! We'll always remember that eventually we will reverse all the steps: $$\begin{aligned} \lvert 2x^2 - 2a^2\rvert &< e \\ 2\lvert x^2 - a^2\rvert &< e \\ 2\lvert(x-a)(x+a)\rvert &< e \\ 2\lvert x-a\rvert * \lvert x+a\rvert &< e \\ \lvert x-a\rvert &< e/(2\lvert x+a\rvert) \end{aligned}$$ *Now, we once again get stuck. We basically have* $d = e/(2\lvert x+a\rvert)$ *which is* in terms of x and a, *but* x *is defined later. So, now we attempt to bound* $\lvert x+a\rvert$. ***It is very important to notice that now, we switch to working forwards!*** *Since we get to choose* d*, we can add the extra restriction that* $d < 1$*. Since* $\lvert x-a\rvert < d < 1$*, we add* $\lvert 2a\rvert$ *to both sides. Now, we have* $\lvert x+a\rvert < \lvert x-a\rvert + \lvert 2a\rvert < 1 + \lvert 2a\rvert$*. Finally, we can choose an* e*.*
Choose $d = \min\left(0.99, \frac{e}{2(1+\lvert 2a\rvert)}\right)$	*Above, we figured out that we needed* $d < 1$ *and* $d < e/(2\lvert x+a\rvert)$*. So, we put those together with a* $\min$*.*
Suppose $\lvert x-a\rvert < d$.	*We need to prove an implication...*

So, $\|x-a\| < 1$ and $\|x-a\| < \frac{e}{2(1+\|2a\|)}$ by definition of d.	*The only thing that we've done in our proof and not used is the definition of d.*

So, multiplying both sides, we have $$2(\|x-a\|(1+\|2a\|)) < e$$	*Our goal is to show $\|2x^2 - 2a^2\| < e$; so, we want to make both sides look more like this... Conveniently, we already know the algebra we want. It's the "reversed work" from above.*
Since $1+\|2a\| => \|x-a\|+\|2a\| > \|x+a\|$, we have $$e > 2(\|x-a\|(1+\|2a\|)) > 2(\|x-a\|\|x+a\|)$$	*Seriously, we've already figured out this argument... look up above. You'll see it almost verbatim!*
Now, note that by distributing and simplifying, we have $e > 2(\|x-a\|\|x+a\|) = 2\|x^2-a^2\| = \|2x^2 - 2a^2\|$, which is what we were trying to prove.	*Seriously, we've already figured out this argument... look up above. You'll see it almost verbatim!*
So, the claim is true!	*Conclude our proof...*