What Is This?

Each of the following is as close as we can get to giving you a template (and a completely worked out example) for every proof technique we will discuss this quarter.

However, there is a large WARNING associated with these templates! It might be tempting to memorize the structure(s) of these templates rather than learn what they mean well enough to duplicate them on your own. DON’T DO IT!!! These are meant as a way to help you ease into proof writing as we introduce more and more complicated strategies. There isn’t (and will never be) an algorithm or formula for writing proofs.

Contents

1 Direct Proofs .................................................. 2
   1.1 Technique Outlines ....................................... 2
   1.2 Example .................................................. 2

2 Implication Proofs ............................................... 3
   2.1 Technique Outlines ....................................... 3
   2.2 Examples .................................................. 4

3 Contradiction Proofs ........................................... 5
   3.1 Technique Outlines ....................................... 5
   3.2 Example .................................................. 5

4 Set Proofs ....................................................... 6
   4.1 Technique Outlines ....................................... 6
   4.2 Example .................................................. 7

5 Induction Proofs ............................................... 8
   5.1 Technique Outlines ....................................... 8
   5.2 Example .................................................. 9

6 Strong Induction Proofs ...................................... 10
   6.1 Technique Outline ........................................ 10
   6.2 Example .................................................. 11

7 Structural Induction Proofs .................................. 13
   7.1 Technique Outline ........................................ 13
   7.2 Example .................................................. 13

8 Irregularity Proofs .......................................... 14
   8.1 Technique Outline ........................................ 14
   8.2 Example .................................................. 14

9 Diagonalization Proofs ....................................... 15
   9.1 Technique Outline ........................................ 15
   9.2 Example .................................................. 15
1 Direct Proofs

1.1 Technique Outlines

Proving a \( \forall \) Statement

Prove \( \forall x \ P(x) \).

Let \( x \) be arbitrary.

Now, \( x \) represents an arbitrary element, and we can just use it.

Prove \( P(x) \) by some other strategy.

Since \( x \) was arbitrary, the claim is true.

Prove \( \forall x \ (x = 5 \lor x \neq 5) \).

Let \( x \) be arbitrary.

Note that by the law of excluded middle, \( x = 5 \) or \( x \neq 5 \).

Since \( x \) was arbitrary, the claim is true.

Proving an \( \exists \) Statement

Prove \( \exists x \ P(x) \).

[Find an \( x \) for which \( P(x) \) is true. This is not actually part of the proof, but it’s necessary to continue.]

Let \( x = \) expression that satisfies \( P(x) \).

Now, explain why \( P(x) \) is true.

Since \( P(x) \) is true, the claim is true.

Prove \( \exists x \ \text{Even}(x) \).

[We can choose any even number here. We’ll go with 2, because it’s simplest.]

Let \( x = 2 \).

Note that 2 is even, by definition, because \( 2 \times 1 = 2 \).

Since 2 is even, the claim is true.

Disproving a Statement

Disprove \( P(x) \).

We show that \( P(x) \) is false by proving its negation: the negation of \( P(x) \).

Prove \( \neg P(x) \) using some other proof strategy.

Since \( \neg P(x) \) is true, \( P(x) \) is false.

Disprove Odd(4).

We show that 4 is not odd by showing it’s even.

Note that 4 is even, by definition, because \( 2 \times 2 = 4 \).

Since 4 is even, it is not odd.

1.2 Example

Prove \( \forall x \ \forall y \ \exists z \ (zx = y) \) Domain: Non-Zero Reals

Proof: Let \( x \) and \( y \) be arbitrary. Choose \( z = \frac{y}{x} \). Note that \( x \times \frac{y}{x} = y \). This is valid, because \( x \neq 0 \). Thus, we’ve found a \( z \) \( (yx) \) such that the claim is true.

Commentary: We started off the proof with “Let \( x \) and \( y \) be arbitrary”. This is so that the claim works for any \( x \) and \( y \) we are provided. We’re not allowed to assume anything special about \( x \) or \( y \), but if we use them as if they are any particular number, the claim will be true for any \( x \) and \( y \). The “choose” line is used to prove the existential quantifier by pointing out a value that works. We have to follow that up with a justification of why the choice we made works. The last line just sums up what we’ve done.
## 2 Implication Proofs

### 2.1 Technique Outlines

#### Proving an \( \implies \) (Directly)

<table>
<thead>
<tr>
<th>Prove an ( A \implies B ).</th>
<th>Prove that if ( x \leq 4 ) is an even, positive integer, then it’s a power of two.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Suppose ( A ) is true.</td>
<td>Suppose ( x \leq 4 ) is even, positive integer.</td>
</tr>
<tr>
<td></td>
<td>Since ( x ) is a positive integer, ( x &gt; 0 ). Furthermore, since ( x \leq 4 ), it must be that ( x = 2 ) or ( x = 4 ). Note that ( 2 = 2^1 ) and ( 4 = 2^2 ); so, both possibilities are powers of two.</td>
</tr>
<tr>
<td>Prove ( B ) using the additional assumption that ( A ) is true.</td>
<td>It follows that ( x ) must be a power of two. So, if ( x ) is an even positive integer at most four, then ( x ) is a power of two.</td>
</tr>
</tbody>
</table>

It follows that \( B \) is true. Therefore, \( A \implies B \).

#### Proving an \( \implies \) (Contrapositive)

<table>
<thead>
<tr>
<th>Prove an ( A \implies B ).</th>
<th>Prove that if ( x^2 - 6x + 9 \neq 0 ), then ( x \neq 3 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>We go by contrapositive. Suppose ( \neg B ) is true.</td>
<td>We go by contrapositive. Suppose ( x = 3 ).</td>
</tr>
<tr>
<td>Prove ( \neg A ) using the additional assumption that ( \neg B ) is true.</td>
<td>Then, ( x^2 - 6x + 9 = 3^2 - 6 \times 3 + 9 = 0. )</td>
</tr>
<tr>
<td>So, ( \neg A ) is true. Therefore, ( A \implies B ).</td>
<td>So, ( x^2 - 6x + 9 = 0 ). Thus, if ( x^2 - 6x + 9 \neq 0 ), then ( x \neq 3 ).</td>
</tr>
</tbody>
</table>
2.2 Examples

Prove $\forall x \forall y \((x + y = 1) \rightarrow (xy = 0))$  

**Domain:** Non-negative Integers

**Proof:** Let $x$ and $y$ be arbitrary non-negative integers.

We prove the implication by contrapositive. Suppose $xy \neq 0$. Then, it must be the case that neither $x$ nor $y$ is zero, because $0 \times a = 0$ for any $a$. So, $x > 0$ and $y > 0$, which is the same as $x \geq 1$ and $y \geq 1$.

Adding inequalities together, we see that $x + y \geq 2$. It follows that $x + y > 1$ which means $x + y \neq 1$ which is what we were trying to show.

So, the original claim is true.

**Commentary:** The hardest thing about proof by contrapositive is to understand when to use it. There are two “clear” situations to try it in:

1. If there are a lot of negations in the statement. (See the example above in the previous section.) Contrapositive adds a bunch of negations into each part of the implication which means if there are already a lot of them, it removes them!

2. If you try the direct proof and get stuck (or feel like you have to use proof by contradiction). A very common mistake is to use proof by contradiction when a proof by contrapositive would be much more clear!

---

Prove $\forall x \forall y \((x < y) \rightarrow (\exists z \ x < z \land z < y))$  

**Domain:** Rationals

**Proof:** Let $x, y$ be arbitrary rational numbers such that $x < y$.

Since $x, y$ are both rational, we have $x = \frac{p_x}{q_x}$ and $y = \frac{p_y}{q_y}$ for integers $p_x, q_x, p_y, q_y$ such that $q_x \neq 0$ and $q_y \neq 0$.

Suppose for contradiction that there are no rationals between $x$ and $y$. Note that $x \neq y$; so, it cannot be the case that $p_x = p_y$ and $q_x = q_y$.

Define $z = \frac{p_z}{q_z} = \frac{p_x + p_y}{2q_xq_y} = \frac{p_xq_y + q_xp_y}{2q_xq_y} = \frac{p_xq_y + p_yq_x}{2q_xq_y}$.

First, note that $p_xq_y + p_yq_x$ is an integer (because it’s a linear combination of integers). Second, note that $2q_xq_y$ is a non-zero integer, because $q_x, q_y \neq 0$.

Furthermore, note that $\frac{p_z}{q_z}$ is the average of $x$ and $y$. Since $x \neq y$, the average must be larger than $x$ and less than $y$.

It follows that $z$ is a rational number such that $x < z < y$, which is what we were trying to prove.

So, the implication is true, as is the entire statement.