CSE 311: Foundations of Computing I

Section : FOL and Inference Solutions

0. Formal Proofs

For this question only, write *formal proofs*.

(a) Prove $\forall x\ (R(x) \land S(x))$ given $\forall x\ (P(x) \to (Q(x) \land S(x)))$, and $\forall x\ (P(x) \land R(x))$.

Solution:

1.	Let x be arbitrary.	
2.	$\forall x\ (P(x) \land R(x))$	[Given]
3.	$P(x) \land R(x)$	[Elim $\forall$: 2]
4.	$P(x)$	[Elim $\land$: 3]
5.	$R(x)$	[Elim $\land$: 3]
6.	$\forall x\ (P(x) \to (Q(x) \land S(x)))$	[Given]
7.	$P(x) \to (Q(x) \land S(x))$	[Elim $\forall$: 6]
8.	$Q(x) \land S(x)$	[MP: 4, 7]
9.	$S(x)$	[Elim $\land$: 8]
10.	$R(x) \land S(x)$	[Intro $\land$: 5, 9]
11.	$\forall x\ (R(x) \land S(x))$	[Intro $\forall$: 10]

(b) Prove $\exists x\ \neg R(x)$ given $\forall x\ (P(x) \lor Q(x))$, $\forall x\ (\neg Q(x) \lor S(x))$, $\forall x\ (R(x) \to \neg S(x))$, and $\exists x\ \neg P(x)$.

Solution:

1.	$\exists x\ \neg P(x)$	[Given]
2.	$\neg P(c)$	[Elim $\exists$: 1]
3.	$\forall x\ (P(x) \lor Q(x))$	[Given]
4.	$P(c) \lor Q(c)$	[Elim $\forall$: 3]
5.	$Q(c)$	[Elim $\lor$: 2, 4]
6.	$\forall x\ (\neg Q(x) \lor S(x))$	[Given]
7.	$\neg Q(c) \lor S(c)$	[Elim $\forall$: 6]
8.	$S(c)$	[Elim $\lor$: 5, 7]
9.	$\forall x\ (R(x) \to \neg S(x))$	[Given]
10.	$R(c) \to \neg S(c)$	[Elim $\forall$: 9]
11.	$\neg\neg S(c) \to \neg R(c)$	[Contrapositive: 10]
12.	$S(c) \to \neg R(c)$	[Double Negation: 11]
13.	$\neg R(c)$	[MP: 8, 12]
14.	$\exists x\ \neg R(x)$	[Intro $\exists$: 13]

1. Odds and Ends

Prove that for any even integer, there exists an odd integer greater than that even integer.

Solution:

Let x be an arbitrary even integer. By the definition of even, we know $x = 2y$ for some corresponding integer y. Now, we define z to be the integer $2y + 1$, which is odd by the definition of odd. By algebra, $2y + 1 > 2y$ regardless of y, so we also know $z > x$. We've now shown that there exists some integer z which is both odd and greater than x. Since x was arbitrary, we can generalize our conclusion to all even integers.

2. Magic Squares

Prove that if a real number $x \neq 0$, then $x^2 + \frac{1}{x^2} \geq 2$.

Solution:

Note that $(x^2 - 1)^2 \geq 0$, because all squares are at least 0. Distributing, we see that $x^4 + 1 \geq 2x^2$. Since $x \neq 0$, we can divide by x^2 to get $x^2 + \frac{1}{x^2} \geq 2$, which is what we were trying to prove.

Note: The first step may seem like "magic", but the way we generally solve these sorts of problems is by working backward and reversing the entire proof.

3. Primality Checking

When brute forcing if the number p is prime, you only need to check possible factors up to $\sqrt{p}$. In this problem, you'll prove why that is the case. Prove that if $n = ab$, then either a or b is at most $\sqrt{n}$.

(*Hint:* You want to prove an implication; so, start by assuming $n = ab$. Then, continue by writing out your assumption for contradiction.)

Solution:

Suppose that $n = ab$. Suppose for contradiction that $a, b > \sqrt{n}$. It follows that $ab > \sqrt{n}\sqrt{n} = n$. We clearly can't have both $n = ab$ and $n < ab$; so, this is a contradiction. It follows that a or b is at most $\sqrt{n}$.

4. Even More Negative

Show that $\forall(x \in \mathbb{Z})$. Even$(x) \to (-1)^x = 1$

Solution:

Let $x \in \mathbb{Z}$ be arbitrary, and suppose x is even. Choose an $n \in \mathbb{Z}$ such that $x = 2n$ (we know such an n exists by the definition of even). It follows that we can rewrite this as

$$\begin{aligned}
(-1)^x &= (-1)^{2n} && [x = 2n] \\
&= ((-1)^2)^n \\
&= (1)^n && [(-1)^2 = 1] \\
&= 1 && [\forall n \in \mathbb{R},\ 1^n = 1]
\end{aligned}$$

We let x be arbitrary, assumed that it was even, and then showed that the implication holds. it follows that for every even integer x, $(-1)^x = 1$. Thus the claim is proven.

5. That's Odd...

Prove that every odd natural number can be expressed as the difference between two consecutive perfect squares.

Solution:

Let $x \in \mathbb{N}$ be arbitrary, and suppose it is odd. Using the definition of odd, choose a $k \in \mathbb{N}$ such that $x = 2k + 1$. Now define n as $k + 1$ (which is also a natural number, since $\mathbb{N}$ is closed under addition), such that $k = n - 1$. Thus, $x = 2k + 1 = 2(n - 1) + 1 = 2n - 1$. Starting with this, we have

$$
\begin{aligned}
x = 2n-1 &= n^2 - n^2 + 2n - 1 && \text{[Add and subtract } n^2\text{]}\\
&= n^2 - (n^2 - 2n + 1) && \text{[rewrite]}\\
&= n^2 - (n-1)^2 && \text{[factoring]}
\end{aligned}
$$

Thus, x can be expressed as the difference between n^2 and $(n-1)^2$, both of which are, by definition, perfect squares. Since x was arbitrary, it follows that any odd integer can be expressed as the difference between two squares.

6. United We Stand

We say that a set S is closed under an operation $\star$ iff $\forall(x, y \in S)\ x \star y \in S$.

(a) Prove $\mathbb{Z}$ is closed under $-$.

Solution:

Let $a, b \in \mathbb{Z}$. We want to show that $(a-b) \in \mathbb{Z}$. By the theorem given, since $b \in \mathbb{Z}$, we know that $-b \in \mathbb{Z}$. Thus, we can rewrite $a-b$ as $a + (-b)$, which is addition between two integers. We are given that $\mathbb{Z}$ is closed under addition, so this number is also an integer. It follows that $\mathbb{Z}$ is closed under subtraction.

(b) Prove that $\mathbb{Z}$ is *not* closed under $/$.

Solution:

Choose $a = 1$ and $b = 2$. Both of these are integers. $\frac{a}{b} = \frac{1}{2} = 0.5$ which is not an integer. Thus, we have provided a counter-example to prove that the integers are *not* closed under division.

(c) Prove that $\mathbb{I}$ is *not* closed under $+$.

Solution:

To prove that $\mathbb{I}$ is not closed under addition, we need to prove the negation of the definition of closure.

We want to show $\exists(x, y \in \mathbb{I})\ x + y \notin \mathbb{I}$. Choose $x = \sqrt{2}$ and choose $y = -\sqrt{2}$. Note that both of these are irrational (we'll prove this later). Also, note that $x + y = \sqrt{2} - \sqrt{2} = 0$. We can write 0 as $\dfrac{0}{1}$. Since 0 and 1 are both integers, this shows that $x + y$ is a rational number by definition of rational numbers. This means that $x + y$ is not irrational, proving the claim that the irrationals are not closed under addition.

7. A Hint of Things to Come

Prove that $\forall(a, b \in \mathbb{Z})$. $a^2 - 4b \neq 2$.

Solution:

Let $a, b \in \mathbb{Z}$ be arbitrary. Assume for the sake of contradiction that $a^2 - 4b = 2$. Then, $a^2 = 4b + 2 = 2(2b+1)$. Since $b \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under addition and multiplication, $(2b+1) \in \mathbb{Z}$. By definition, this means a^2 is even. We have shown in class that $\forall(a \in \mathbb{Z})$ Even$(a^2) \rightarrow$ Even(a), so we know that a is even. By definition, there exists a $k \in \mathbb{Z}$ such that $a = 2k$. Choose such a k.

We can replace a with $2k$ in the original statement to get $(2k)^2 - 4b = 2 \leftrightarrow 2k^2 - 2b = 1 \leftrightarrow 2(k^2 - b) = 1$. Since $k, b \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under multiplication and addition, this means that $(k^2 - b) \in \mathbb{Z}$. It follows by the definition of even numbers that 1 is even. However, we know that 1 is in fact odd (choose $k = 0$, then $1 = 2k + 1$, the definition of odd). Thus, we have found a contradiction.

It follows that our assumption is false, so the original statement must be true.