CSE 311

Foundations of Computing I

* All slides are a combined effort between previous instructors of the course

Administrivia

Token verifications should have been e-mailed to you!

The midterm will be on Wed, May 4 from 4:30pm – 6:00pm in JHN 102

If you cannot make this time, and you haven't already e-mailed me, you need to tell me **right after lecture**.

There will be two review sessions:

- Saturday from 1pm – 3pm in EEB 105
- Tuesday from 4:30pm – 6:30pm in EEB 105

Prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

Let $P(n)$ be "$3 \mid 2^{2n} - 1$". We go by induction on n.

Base Case (n=0): Note that $2^{2 \cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$.

We know $3 \mid 0$, by definition of divides, because $3 \cdot 0 = 0$. So, P(0) is true.

Induction Hypothesis: Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step: We want to show $P(k+1)$. That is, WTS $3 \mid 2^{2(k+1)} - 1$.

Note that $2^{2(k+1)} - 1 = 2^{2k+2} - 1$ [Algebra]

$= (2^{2k})(2^2) - 1$ [Algebra]

$= (2^{2k} - 1 + 1)(2^2) - 1$ [Algebra]

By IH, we know $3 \mid 2^{2k} - 1$. So, by definition of divides, we know $2^{2k} - 1 = 3j$ for some j.

$= (3j + 1)(4) - 1 = 3(4j + 1)$ [Algebra]

So, by definition of divides, $3 \mid 2^{2(k+1)} - 1$.

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.

So, the claim is true for all natural numbers by induction.

We know (by IH)...

$3 \mid 2^{2k} - 1$

...which means...

$2^{2k} - 1 = 3j$

We're trying to get...

$3 \mid 2^{2(k+1)} - 1$

...which is true if...

$2^{2(k+1)} - 1 = 3k$

Prove $3^n \geq n^2$ for all $n \geq 3$.

Let $P(n)$ be "$3^n \geq n^2$". We go by induction on n.

Base Case (n=3): Note that $3^3 = 27 \geq 9 = 3^2$. So, P(3) is true.

Induction Hypothesis: Suppose P(k) is true for some $k \geq 3$.

Induction Step: We want to show P($k + 1$).

Note that
$$
\begin{aligned}
3^{k+1} &= 3(3^k) && \text{[Algebra]}\\
&\geq 3(k^2) && \text{[By IH]}\\
&= k^2 + k \cdot k + k^2 && \text{[Algebra]}\\
&\geq k^2 + 2 \cdot k + k^2 && [k \geq 2]\\
&\geq k^2 + 2 \cdot k + 1^2 && [k \geq 1]\\
&\geq k^2 + 2k + 1
\end{aligned}
$$

This is exactly P($k + 1$). So, P(k) $\rightarrow$ P($k + 1$).

So, the claim is true for all $n \geq 3$ by induction.

We know (by IH)...

$$3^k \geq k^2$$

We're trying to get...

$$3^{k+1} \geq (k+1)^2$$
$$= k^2 + 2k + 1$$

Prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Let $P(n)$ be "$2n^3 + 2n - 5 \geq n^2$". We go by induction on n.

Base Case (n=2): Note that $2(2^3) + 2(2) - 5 = 15 \geq 4 = 2^2$

Induction Hypothesis: Suppose the claim is true for some $k \geq 2$.

Induction Step: We want to show $P(k+1)$.

Note that $2(k+1)^3 + (2k+1) - 5 = 2(k+1)(k^2 + 2k + 1) + (2k+1) - 5$

[Algebra]

$$= 2(k^3 + 2k^2 + k + k^2 + 2k + 1) + (2k+1) - 5$$
$$= 2k^3 + 4k^2 + 2k + 2k^2 + 4k + 2 + (2k+1) - 5$$
$$= 2k^3 + 6k^2 + 6k + 2 + (2k+1) - 5$$
$$= (2k^3 + 2k - 5) + 6k^2 + 6k + 3$$

[By IH]

$$\geq k^2 + 6k^2 + 6k + 3 = 7k^2 + 6k + 3$$

[Algebra]

$$= (k^2 + 2k + 1) + 6k^2 + 4k + 3$$
$$= (k+1)^2 + 6k^2 + 4k + 3$$

[k ≥ 2]

$$\geq (k+1)^2$$

This is exactly $P(k+1)$. So, $P(k) \to P(k+1)$.

So, the claim is true for all $n \geq 2$ by induction.

We know (by IH)...

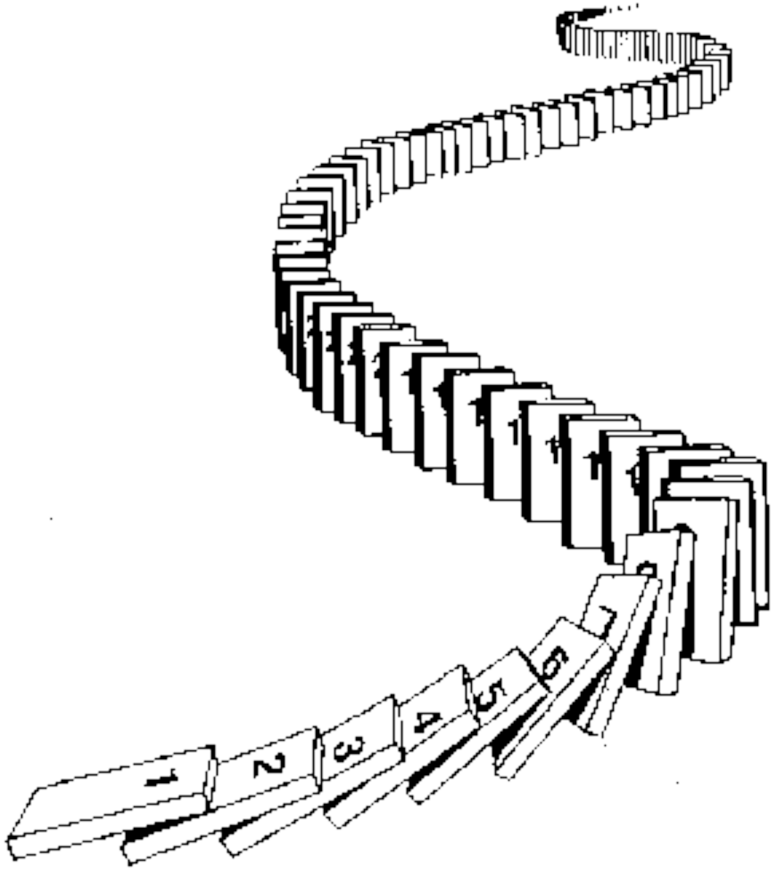
$$2k^3 + 2k - 5 \geq k^2$$

We're trying to get...

$$2(k+1)^3 + 2(k+1) - 5 \geq (k+1)^2$$
$$(k+1)^2 = k^2 + 2k + 1$$

CSE 311: Foundations of Computing

Lecture 15: Strong Induction

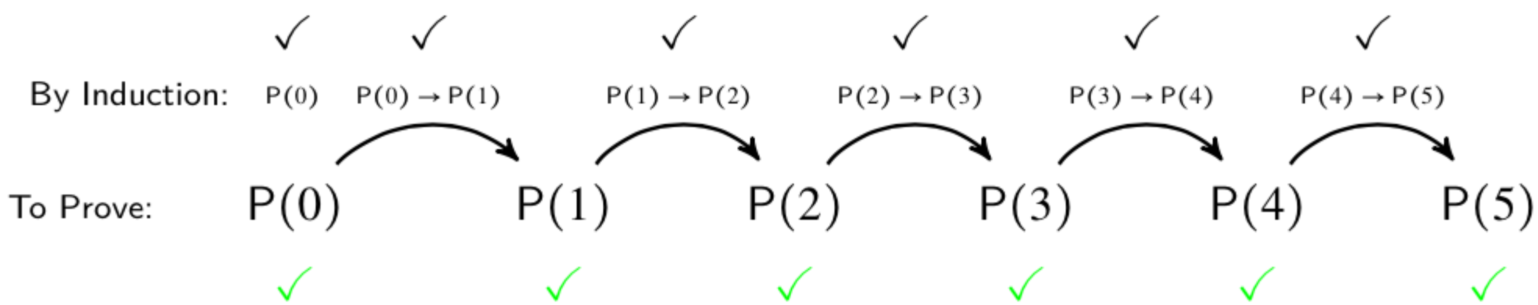


Induction Is A Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall\, k\, (P(k) \rightarrow P(k+1))$$
$$\therefore\ \forall\, n\, P(n)$$

How does this technique prove P(5)?



First, we prove P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Domain: Natural Numbers

Induction Is A Rule of Inference

"Induction"

1.	P(0)	("Given")
2.	$\forall n\ (\mathrm{P}(n) \rightarrow \mathrm{P}(n+1))$	("Given")
3.	P(1)	(MP: 2, 1)
4.	P(2)	(MP: 2, 3)
5.	P(3)	(MP: 2, 4)
6.	P(4)	(MP: 2, 5)

Notice how when we use regular induction, we're already proving the things necessary to use strong induction.

This is no extra work with a benefit!

"Strong Induction"

1.	P(0)	("Given")
2.	$\forall n\ ((\mathrm{P}(0) \wedge \mathrm{P}(1) \wedge \cdots \wedge \mathrm{P}(n) \rightarrow \mathrm{P}(n+1))$	("Given")
3.	P(1)	(MP: 2, 1)
4.	P(2)	(MP: 2, 1, 3)
5.	P(3)	(MP: 2, 1, 3, 4)
6.	P(4)	(MP: 2, 1, 3, 4, 5)

Strong Induction

$$P(0)$$

$$\forall k\ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \to P(k+1)\Big)$$

$$\therefore \forall n\ P(n)$$

Strong Induction English Proof

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

Every $n \geq 2$ can be expressed as a product of primes.

Let $P(n)$ be "$n = p_0 p_1 \cdots p_j$, where $p_0, p_1, \ldots, p_j$ are prime."

We go by induction on n.

Base Case (n=2): Note that 2 is prime (which means it's a product of primes).

Induction Hypothesis: Suppose that P(2), P(3), …, P(k – 1) are true for some $k \geq 2$.

Induction Step: We go by cases.

Case 1 (k is prime):

Then, since k is prime, k is a product of primes.

Case 2 (k is composite):

Then, by definition of composite, we have non-trivial $1 < a, b < k$ such that k = ab. Since a and b are between 2 and k – 1, we know P(2) and P(k – 1) are true. So, we have:

$$a = p_0 p_1 \cdots p_j \text{ and } b = p_{j+1} p_{j+2} \cdots p_{j+\ell}$$

Then, $k = ab = p_0 p_1 \cdots p_j p_{j+1} p_{j+2} \cdots p_{j+\ell}$

So, k can be expressed as a product of primes.

So, P(n) is true for all $n \geq 2$ is true by induction.

We know (by IH)…

All numbers smaller than k can be expressed as a product of primes.

We're trying to get…

k can be expressed as a product of primes.