Adam Blank — Spring 2016

CSE 311

Foundations of Computing I

* All slides are a combined effort between previous instructors of the course

Administrivia

Token verifications should have been e-mailed to you!

The midterm will be on Wed, May 4 from 4:30pm – 6:00pm in JHN 102

If you cannot make this time, and you haven't already e-mailed me, you need to tell me **right after lecture**.

There will be two review sessions:

- Saturday from 1pm – 3pm in EEB 105
- Tuesday from 4:30pm – 6:30pm in EEB 105

Prove $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

Let $P(n)$ be "$3 \mid 2^{2n} - 1$". We go by induction on n.

Base Case (n=0): Note that $2^{2\cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$.
We know $3 \mid 0$, by definition of divides, because $3 \cdot 0 = 0$. So, P(0) is true.

Induction Hypothesis: Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step: We want to show $P(k+1)$. That is, WTS $3 \mid 2^{2(k+1)} - 1$.

Note that $2^{2(k+1)} - 1 = 2^{2k+2} - 1$ [Algebra]

$= (2^{2k})(2^2) - 1$ [Algebra]

$= (2^{2k} - 1 + 1)(2^2) - 1$ [Algebra]

By IH, we know $3 \mid 2^{2k} - 1$. So, by definition of divides, we know $2^{2k} - 1 = 3j$ for some j.

$= (3j+1)(4) - 1 = 3(4j+1)$ [Algebra]

So, by definition of divides, $3 \mid 2^{2(k+1)} - 1$.

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.
So, the claim is true for all natural numbers by induction.

We know (by IH)...
$3 \mid 2^{2k} - 1$
...which means...
$2^{2k} - 1 = 3j$

We're trying to get...
$3 \mid 2^{2(k+1)} - 1$
...which is true if...
$2^{2(k+1)} - 1 = 3k$

Prove $3^n \geq n^2$ for all $n \geq 3$.

Let $P(n)$ be "$3^n \geq n^2$". We go by induction on n.

Base Case (n=3): Note that $3^3 = 27 \geq 9 = 3^2$. So, P(3) is true.

Induction Hypothesis: Suppose $P(k)$ is true for some $k \geq 3$.

Induction Step: We want to show $P(k+1)$.

Note that $3^{k+1} = 3(3^k)$ [Algebra]

$\geq 3(k^2)$ [By IH]

$= k^2 + k \cdot k + k^2$ [Algebra]

$\geq k^2 + 2 \cdot k + k^2$ [k ≥ 2]

$\geq k^2 + 2 \cdot k + 1^2$ [k ≥ 1]

$\geq k^2 + 2k + 1$

This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.
So, the claim is true for all $n \geq 3$ by induction.

We know (by IH)...
$3^k \geq k^2$

We're trying to get...
$3^{k+1} \geq (k+1)^2$
$= k^2 + 2k + 1$

Prove $2n^3 + 2n - 5 \geq n^2$ for all $n \geq 2$.

Let $P(n)$ be "$2n^3 + 2n - 5 \geq n^2$". We go by induction on n.

Base Case (n=2): Note that $2(2^3) + 2(2) - 5 = 15 \geq 4 = 2^2$

Induction Hypothesis: Suppose the claim is true for some $k \geq 2$.

Induction Step: We want to show $P(k+1)$.

Note that $2(k+1)^3 + (2k+1) - 5 = 2(k+1)(k^2+2k+1) + (2k+1) - 5$

[Algebra]

$= 2(k^3 + 2k^2 + k + k^2 + 2k + 1) + (2k+1) - 5$

$= 2k^3 + 4k^2 + 2k + 2k^2 + 4k + 2 + (2k+1) - 5$

$= 2k^3 + 6k^2 + 6k + 2 + (2k+1) - 5$

$= (2k^3 + 2k - 5) + 6k^2 + 6k + 3$

[By IH] $\geq k^2 + 6k^2 + 6k + 3 = 7k^2 + 6k + 3$

$= (k^2 + 2k + 1) + 6k^2 + 4k + 3$

[Algebra] $= (k+1)^2 + 6k^2 + 4k + 3$

[k ≥ 2] $\geq (k+1)^2$

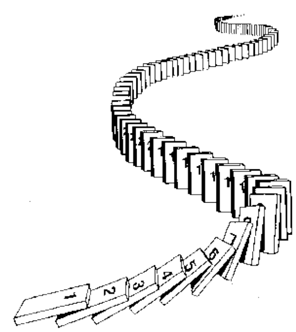
This is exactly $P(k+1)$. So, $P(k) \rightarrow P(k+1)$.
So, the claim is true for all $n \geq 2$ by induction.

We know (by IH)...
$2k^3 + 2k - 5 \geq k^2$

We're trying to get...
$2(k+1)^3 + 2(k+1) - 5 \geq (k+1)^2$
$(k+1)^2 = k^2 + 2k + 1$

CSE 311: Foundations of Computing

Lecture 15: Strong Induction



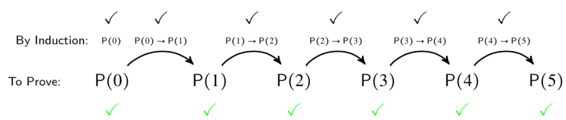
Induction Is A Rule of Inference

Domain: Natural Numbers

$P(0)$
$\forall\ k\ (P(k)\ \rightarrow\ P(k+1))$

$\therefore\ \forall\ n\ P(n)$

How does this technique prove P(5)?



First, we prove P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Induction Is A Rule of Inference

Domain: Natural Numbers

"Induction"

1. P(0) ("Given")
2. $\forall n\ (P(n) \rightarrow P(n+1))$ ("Given")
3. P(1) (MP: 2, 1)
4. P(2) (MP: 2, 3)
5. P(3) (MP: 2, 4)
6. P(4) (MP: 2, 5)

Notice how when we use regular induction, we're already proving the things necessary to use strong induction.

This is no extra work with a benefit!

"Strong Induction"

1. P(0) ("Given")
2. $\forall n\ ((P(0) \wedge P(1) \wedge \cdots \wedge P(n) \rightarrow P(n+1))$ ("Given")
3. P(1) (MP: 2, 1)
4. P(2) (MP: 2, 1, 3)
5. P(3) (MP: 2, 1, 3, 4)
6. P(4) (MP: 2, 1, 3, 4, 5)

Strong Induction

$P(0)$
$\forall k\ \big((P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \rightarrow P(k+1)\big)$

$\therefore \forall n\ P(n)$

Strong Induction English Proof

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis: Assume that for some arbitrary integer $k\ \geq\ 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step: Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

Every $n \geq 2$ can be expressed as a product of primes.

Let $P(n)$ be "$n = p_0 p_1 \cdots p_j$, where $p_0, p_1, \ldots, p_j$ are prime."

We go by induction on n.

Base Case (n=2): Note that 2 is prime (which means it's a product of primes).

Induction Hypothesis: Suppose that P(2), P(3), …, P(k – 1) are true for some $k \geq 2$.

Induction Step: We go by cases.

Case 1 (k is prime):
Then, since k is prime, k is a product of primes.

Case 2 (k is composite):
Then, by definition of composite, we have non-trivial $1 < a, b < k$ such that k = ab. Since a and b are between 2 and k – 1, we know P(2) and P(k – 1) are true. So, we have:

$a = p_0 p_1 \cdots p_j$ and $b = p_{j+1} p_{j+2} \cdots p_{j+\ell}$

Then, k = ab = $p_0 p_1 \cdots p_j p_{j+1} p_{j+2} \cdots p_{j+\ell}$

So, k can be expressed as a product of primes.

So, P(n) is true for all $n \geq 2$ is true by induction.

We know (by IH)…

All numbers smaller than k can be expressed as a product of primes.

We're trying to get…

k can be expressed as a product of primes.