Adam Blank Spring 2016

CSE 311

Foundations of Computing I

* All slides are a combined effort between previous instructors of the course

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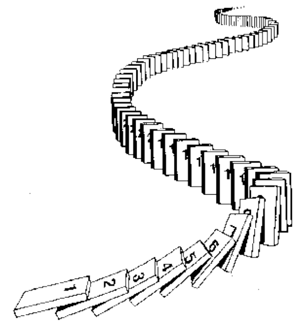
CSE 311: Foundations of Computing

Lecture 16: Recursively Defined Sets



Strong Induction

$P(0)$

$\forall k \left(\left(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\right) \to P(k+1) \right)$

$\therefore \forall n\, P(n)$

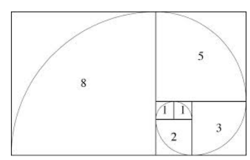
1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. <u>Base Case</u>: Prove $P(0)$
3. <u>Inductive Hypothesis:</u> Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. <u>Inductive Step:</u> Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values □ k)
5. <u>Conclusion:</u> Result follows by induction

Fibonacci Numbers

$f_0 = 0$

$f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$



Bounding the Fibonacci Numbers

Define f_n as: $f_0 = 0$
$f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Theorem:
$2^{n/2-1} \leq f_n$ and $f_n < 2^n$ for all $n \geq 2$

Proof:
Let P(n) be "$2^{n/2-1} \leq f_n$ and $f_n < 2^n$" for all $n \geq 2$.
We go by strong induction on n.

Base Case: $2^{2/2-1} = 2^0 = 1 \leq 0 + 1 = f_2$, and
$f_2 = 0 + 1 = 1 < 4 = 2^2$. So, P(2) is true.

Induction Hypothesis:
Suppose P(j) for all integers j s.t. $2 \leq j \leq k$ for some $k \geq 2$.

Induction Step: We want to show $2^{(k+1)/2-1} \leq f_{k+1}$ and $f_{k+1} < 2^n$

Bounding the Fibonacci Numbers

Define f_n as: $f_0 = 0$
$f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Theorem:
$2^{n/2-1} \leq f_n$ and $f_n < 2^n$ for all $n \geq 2$

Induction Step: We want to show $2^{(k+1)/2-1} \leq f_{k+1}$ and $f_{k+1} < 2^n$

If k+1=3, $2^{3/2-1} = 2^{1/2} \leq 2 = 1 + 1 = f_3$, and
$f_3 = 1 + 1 = 2 < 8 = 2^3$. So, P(3) is true.

Otherwise, note that $f_{k+1} = f_k + f_{k-1}$ by definition.

Taking each inequality separately:

$f_{k+1} = f_k + f_{k-1} < 2^k + 2^{k-1}$ (by IH)
$< 2^k + 2^k$ ($2^{k-1} < 2^k$)
$= 2^{k+1}$

$f_{k+1} = f_k + f_{k-1} \geq 2^{k/2-1} + 2^{(k-1)/2-1}$ (by IH)
$\geq 2^{(k-1)/2-1} + 2^{(k-1)/2-1}$ (Because $2^{k/2-1} > 2^{(k-1)/2-1}$)
$= 2(2^{(k-1)/2-1})$ (Combining terms)
$= 2^{2/2+(k-1)/2-1}$ (Multiplying)
$= 2^{(k+1)/2-1}$

So, the claim is true by strong induction.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for gcd(a,b) with a > b. Then, $a \geq f_{n+1}$.**

We go by strong induction on n.
Let P(n) be "gcd(a,b) with a > b takes n steps → $a \geq f_{n+1}$" for all n ≥ 1.

Base Case:
If Euclid's Algorithm on a, b, with a > b, takes 1 step, then it must be the case that b | a.
Note that $f_2 = 1$.
Note that if a were 0, then gcd(0, b), which takes zero steps. So, the smallest possible value for a is 1, which is f_2.
Induction Hypothesis: Suppose P(j) for all integers j s.t. 1 ≤ j ≤ k for some k ≥ 1.

Induction Step: We want to show if gcd(a,b) takes k+1 steps, then $a \geq f_{k+2}$.
If k = 2, note that a > 1, because gcd(1, b) takes one step. Also, $f_3 = 2$.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for gcd(a,b) with a > b. Then, $a \geq f_{n+1}$.**

Since the algorithm took k+1 steps, let's give them names:

Say $r_{k+1} = a$ and $r_k = b$, and $r_i = r_{i-1} \bmod r_{i-2}$.
So, gcd(a, b) = gcd(r_{k+1}, r_k)
= gcd(r_k, $r_k \bmod r_{k+1}$) = gcd(r_k, r_{k-1})
= gcd(r_{k-1}, $r_{k-1} \bmod r_k$) = gcd(r_{k-1}, r_{k-2})
= …

Writing these as equations, we have:

$r_{k+1} = q_k r_k + r_{k-1}$
$r_k = q_{k-1} r_{k-1} + r_{k-2}$
…
$r_3 = q_2 r_2 + r_1$
$r_2 = q_1 r_1$

Note that $q_i \geq 1$, $r_i \geq 1$.

Note that after one iteration of the algorithm, we're left with gcd(r_k, r_{k-1}) which takes k steps.
By the IH, $r_k \geq f_{k+1}$. So,

$r_{k+1} = q_k r_k + r_{k-1}$ (by gcd algorithm)
$\geq q_k f_{k+1} + f_k$ (by IH)
$\geq f_{k+1} + f_k$ ($q_k \geq 1$)
$\geq f_{k+2}$ (definition of f)

Recursive Definition of Sets

Recursive Definition

- **Basis Step: 0 ∈ S**
- **Recursive Step: If x ∈ S, then x + 2 ∈ S**
- **Exclusion Rule: Every element in S follows from basis steps and a finite number of recursive steps.**

Recursive Definitions of Sets

Basis: 6 ∈ S, 15 ∈ S
Recursive: If x,y ∈ S, then x+y ∈ S

Basis: [1, 1, 0] ∈ S, [0, 1, 1] ∈ S
Recursive: If [x, y, z] ∈ S, then [αx, αy, αz] ∈ S
If [x_1, y_1, z_1] ∈ S and [x_2, y_2, z_2] ∈ S, then [$x_1 + x_2$, $y_1 + y_2$, $z_1 + z_2$] ∈ S.

Powers of 3:
Basis: 1 ∈ S
Recursive: If x ∈ S, then 3x ∈ S.