CSE 311

Foundations of Computing I

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "The square of every even number is even."

Formal proof of: $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$

1.	Let a be arbitrary	Defining a
2.1.	$\text{Even}(a)$	Assumption
2.2.	$\exists y\ (a = 2y)$	Definition of Even by 2.1
2.3.	$a = 2c$	$\exists$ Elim: 2.2
2.4.	$a^2 = 4c^2 = 2(2c^2)$	Algebra
2.5.	$\exists y\ (a^2 = 2y)$	$\exists$ Intro: 2.4
2.6.	$\text{Even}(a^2)$	Definition of Even by 2.5
2.	$\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$	Direct Proof Rule

Even and Odd

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Domain of Discourse
Integers

Initialize variables.
[Header/Intro of the proof]

Let a be an arbitrary even number.

Explain why a^2 is even.
[Body of the proof]

Then, a = 2c for some c, by definition of even.
Squaring both sides, we see $a^2 = 4c^2 = 2(2c^2)$.

Conclude the sub-proof
[“Return” “Inner Result”]

It follows that a^2 is even by definition of even.

Conclude the proof
[“What have we shown?”]

Since a was arbitrary, we’ve shown the square of every even number is even.

Now, Prove “The square of every odd number is odd.”

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "The square of every odd number is odd."

$\cancel{\forall x}\ (\text{Odd}(x) \to \text{Odd}(x^2))$

Let a be arbitrary. Suppose a is odd.

Let a be an arbitrary odd integer.

By def. of odd, $a = 2c+1$ for some int. c. Look $a^2 = (2c+1)^2$
$= 4c^2 + 4c + 1$
$= 2(2c^2 + 2c) + 1$

So, we have found an int b s.t. $a^2 = 2b+1$. So, a^2 is odd. So, the claim is true.

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "The square of every odd number is odd."

Let x be an arbitrary odd number.

Then, $x = 2k+1$ for some integer k (depending on x).

Therefore, $x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Since $2k^2+2k$ is an integer, x^2 is odd.

Counterexamples

To *disprove* ∀x P(x) prove ¬∀x P(x) :

- $\neg\forall x\, P(x) \equiv \exists x \neg P(x)$
- To prove the existential, find an x for which P(x) is **false**
- This example is called a **counterexample.**

Prove blah

Disprove blah

Counterexample…example

Disprove "Every non-negative integer has another number smaller than it."

$$\forall x\, \exists y\, (y < x)$$

Tell the reader that we're about to use a "counterexample".

We claim $\forall x\, \exists y\, (y < x)$ is false. So, we show the negation, $\exists x\, \forall y\, (y \geq x)$, is true.

Use $\exists$ ~~Elim~~ intro.

Let $x = \boxed{0}$

Use $\forall$ ~~Elim~~ intro.

Let y be arb.

Prove the $\forall$ statement.

Since y is a non-neg. int.
So, $y \geq 0 = x$.

Conclude the proof.

So, the claim is true.

Counterexample…example

Disprove "Every non-negative integer has another number smaller than it."

$$\forall x\ \exists y\ (y < x)$$

Tell the reader that we're about to use a "counterexample".

We claim $\forall x\ \exists y\ (y < x)$ is false. So, we show the negation, $\exists x\ \forall y\ (y \geq x)$, is true.

Use ∃ ~~Elim~~ intro.

Consider x = 0.

Use ∀ ~~Elim~~ intro.

Let y be arbitrary.

Prove the ∀ statement.

Since y is non-negative, $y \geq 0$. So, the claim is true.

Conclude the proof.

Thus, the original claim is false.

Reminder for HW

For ~~Intro~~ elim ∃…

Your "c" has to be new **(e. g. cannot be used previously in the proof)**

You should say what variables your "c" depends on.

The order you use Elim ∃ and Elim ∀ in DOES matter!

Reminder: ∃x ∀y P(x,y) IS DIFFERENT FROM ∀y∃x P(x,y)

Proof by Contrapositive: One Strategy for implications

If we assume $\neg q$ and derive $\neg p$, then we have proven $\neg q \rightarrow \neg p$, which is the same as $p \rightarrow q$.

1.1. $\neg q$ Assumption

...

1.3. $\neg p$

1. $\neg q \rightarrow \neg p$ Direct Proof Rule

2. $p \rightarrow q$ Contrapositive: 1

$(p \rightarrow q) \equiv \boxed{(\neg q \rightarrow \neg p)}$

Proof by Contradiction: One way to prove ¬p

If we assume p and derive F (a contradiction), then we have proven ¬p.

prove ¬p.

1.1. p Assumption

...

1.3. F

1. p → F Direct Proof rule
2. ¬p ∨ F Law of Implication: 4
3. ¬p Identity: 5

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "No integer is both even and odd."

English proof: $\neg \exists x\ (\text{Even}(x) \land \text{Odd}(x))$

$\equiv \cancel{\forall x} \neg(\text{Even}(x) \land \text{Odd}(x))$

Let x be arbitrary. Suppose x is even and odd.

By def. of even, $x = 2a$ for some a.

By def. of odd, $x = 2b+1$ for some b.

So, $2a = 2b+1$. $a - b = \frac{1}{2}$. So, $\frac{1}{2}$ is an integer. This is a contradiction.

So, x is not even and odd.

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "No integer is both even and odd."

English proof: $\neg \exists x\ (\text{Even}(x) \wedge \text{Odd}(x))$

$\equiv \forall x\ \neg(\text{Even}(x) \wedge \text{Odd}(x))$

We go by contradiction. Let x be any integer and suppose that it is both even and odd. Then x=2k for some integer k and x=2m+1 for some integer m. Therefore 2k=2m+1 and hence k=m+½.

But two integers cannot differ by ½ so this is a contradiction. So, no integer is both even and odd.

∎

Rational Numbers

Domain of Discourse
Real Numbers

- **A real number x is *rational* iff there exist integers p and q with q≠0 such that x=p/q.**

$$\text{Rational}(x) \equiv \exists p\, \exists q\ ((x=p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge q\neq 0)$$

$\cancel{\frac{p}{0}}$ $\frac{p}{q}$

$\frac{3}{2}$, $\frac{5}{1}$ vs. π, $\sqrt{2}$,

Rationality

Domain of Discourse
Reals → ~~Integers~~

Predicate Definitions

Rational(x) ≡ $\exists p\ \exists q\ ((x = \frac{p}{q} \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge q \neq 0))$

Prove: "If x and y are rational then xy is rational."

Let x, y be arbitrary rationals.

So $x = \frac{p_x}{q_x}$ and $y = \frac{p_y}{q_y}$, where p_x, q_x, p_y, q_y are ints. and $q_x, q_y \neq 0$.

So, $xy = \frac{p_x p_y}{q_x q_y}$ by mult. fractions.

Note $p_x p_y$ is an int, $q_x q_y$ is a non-zero int (since $q_x \neq 0$ and $q_y \neq 0$).

So, xy is rational.

Rationality

Domain of Discourse
Integers

Predicate Definitions
Rational(x) $\equiv \exists p\, \exists q\ ((x = \frac{p}{q} \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge q \neq 0$

Prove: "If x and y are rational then xy is rational."

Let x and y be rational numbers. Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

Note that xy = (ac)/(bd).

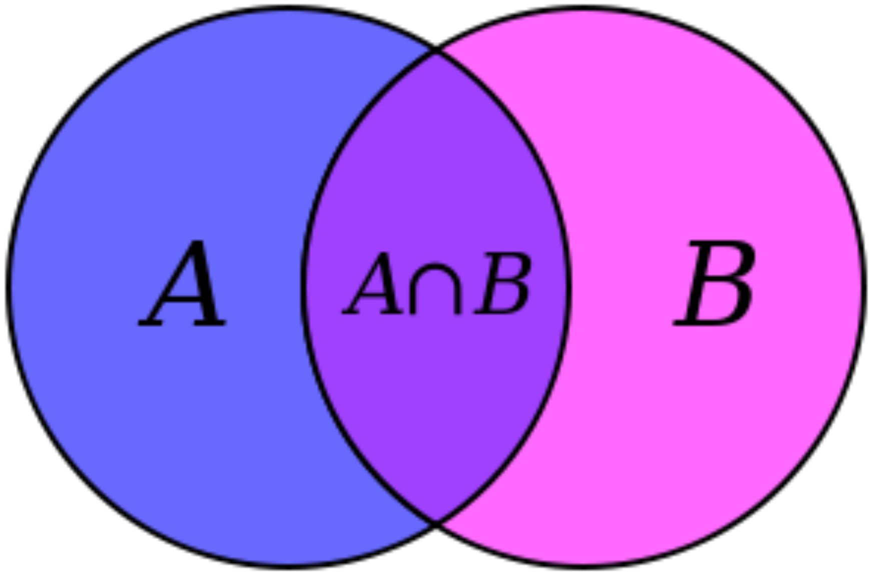
Since b and d are both non-zero, so is bd; furthermore, ac and bd are integers. It follows that xy is rational, by definition of rational.

Proofs

- Formal proofs follow simple well-defined rules and should be easy to check
  - In the same way that code should be easy to execute
- English proofs correspond to those rules but are designed to be easier for humans to read
  - Easily checkable in principle
- Simple proof strategies already do a lot
  - Later we will cover a specific strategy that applies to loops and recursion (mathematical induction)

CSE 311: Foundations of Computing

Lecture 9: Set Theory



Sets

- Mathematical sets are a lot like Java sets:
  - `Set<T> s = new HashSet<T>();`
  - ...with the following exceptions:
    - They are untyped: {"string", 123, 1.2} is a valid set
    - They are immutable: you can't add/remove from them
    - They are built differently
    - They have one fundamental operation:
      - Contains: $x \in S$

Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers**; N = {0, 1, 2, …}

$\mathbb{Z}$ is the set of **Integers**; Z = {…, -2, -1, 0, 1, 2, …}

$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48

$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, π

[n] is the set {1, 2, …, n} when n is a natural number

{} = ∅ is the **empty set**; the *only* set with no elements

EXAMPLES

Are these sets?

A = {1, 1} = {1}

B = {1, 3, 2}

C = {□, 1}

D = {{}, 17}

E = {1, 2, 7, cat, dog, ∅, α}

We say $2 \in E$; $3 \notin E$.

S.add(1)

S.add(1)

Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers;** N = {0, 1, 2, …}
$\mathbb{Z}$ is the set of **Integers**; Z = {…, -2, -1, 0, 1, 2, …}
$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48
$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, π
[n] is the set {1, 2, …, n} when n is a natural number
{} = ∅ is the **empty set**; the *only* set with no elements

We say 2 $\in$ E; 3 $\notin$ E.

EXAMPLES
Are these sets?
A = {1, 1}
B = {1, 3, 2}
C = {□, 1}
D = {{}, 17}
E = {1, 2, 7, cat, dog, ∅, α}

They're all sets.
Note {1} = {1, 1}.

Definition: Equality

A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\,(x \in A \leftrightarrow x \in B)$$

```
boolean equal(Set A, Set B) {
    boolean result = true;
    for (x : A) {
        if (x ∉ B) { result = false; }
    }
    for (x : B) {
        if (x ∉ A) { result = false; }
    }
    return result;
}
```

A = {4, 3, 3}
B = {3, 4, 3}
C = {3, 4}

Are any of A, B, C equal?

They all are!
(dups, order don't matter!)

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall\, x\, (x \in A \rightarrow x \in B)$$

```
boolean subset(Set A, Set B) {
    boolean result = true;
    for (x : A) {
        if (x ∉ B) { result = false; }
    }
    return result;
}
```

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}

$\{\emptyset\}$

QUESTIONS

$\emptyset \subseteq A$?

$\emptyset \in A$

$A \subseteq B$?

$C \subseteq B$?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\,(x \in A \rightarrow x \in B)$$

```
boolean subset(Set A, Set B) {
    boolean result = true;
    for (x : A) {
        if (x ∉ A) { result = false; }
    }
    return result;
}
```

$A = \{1, 2, 3\}$
$B = \{3, 4, 5\}$
$C = \{3, 4\}$

QUESTIONS

$\varnothing \subseteq A$? **Yes. In fact, $\varnothing \subseteq X$ for any set X.**

$A \subseteq B$? **No. $3 \in A$, but that's not true for B.**

$C \subseteq B$? **Yes, $3 \in B$, $4 \in B$.**

Definitions

- A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\,(x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\,(x \in A \rightarrow x \in B)$$

- Note: $\boxed{(\boldsymbol{A} = \boldsymbol{B}) \equiv (\boldsymbol{A} \subseteq \boldsymbol{B}) \wedge (\boldsymbol{B} \subseteq \boldsymbol{A})}$

Building Sets from Predicates

- The following says "S is the set of all x's where P(x) is true.

$$\boxed{S = \{x : P(x)\}}$$

- The following says "those elements of ~~S~~ A for which P(x) is true."

$$\boxed{S = \{x \in A : P(x)\}}$$

- "All the real numbers less than one."
  - $\{x \in \mathbb{R} : x < 1\} = \{x : x \in \mathbb{R} \wedge x < 1\}$
- "All the powers of two that happen to be odd."
  - $\{x \in \mathbb{N} : \exists n (x = 2n+1) \wedge \exists k (x = 2^k)\}$
- "All natural numbers between 1 and n" ("brackets n")
  -

Building Sets from Predicates

- The following says "S is the set of all x's where P(x) is true.

$$S = \{x : P(x)\}$$

- The following says "those elements of S for which P(x) is true."

$$S = \{x \in A : P(x)\}$$

- "All the real numbers less than one"
  - $\{x \in \mathbb{R} : x < 1\}$
- "All the powers of two that happen to be odd."
  - $\{x \in \mathbb{N}: \exists k\ (x = 2k+1) \wedge \exists j\ (x = 2^j)\}$
- "All natural numbers between 1 and n" ("brackets n")
  - $[n] = \{x \in \mathbb{N} : 1 \leq x \leq n\}$

Set Operations

$$A \cup B = \{ x : (x \in A) \vee (x \in B) \}$$ **Union**

$$A \cap B = \{ x : (x \in A) \wedge (x \in B) \}$$ **Intersection**

$$A \setminus B = \{ x : (x \in A) \wedge (x \notin B) \}$$ **Set Difference**

A = {1, 2, 3}
B = {4, 5, 6}
C = {3, 4}

{C}

QUESTIONS

Using A, B, C and set operations, make…

[6] = $A \cup B \cup C$

{3} =

{1,2} =

Set Operations

$$A \cup B = \{ x : (x \in A) \vee (x \in B) \}$$ **Union**

$$A \cap B = \{ x : (x \in A) \wedge (x \in B) \}$$ **Intersection**

$$A \setminus B = \{ x : (x \in A) \wedge (x \notin B) \}$$ **Set Difference**

A = {1, 2, 3}
B = {4, 5, 6}
C = {3, 4}

QUESTIONS

Using A, B, C and set operations, make…

[6] = A ∪ B = A ∪ B ∪ C

{3} = C ∖ B = A ∖ B = A ∩ B

{1,2} = A ∖ C = (A ∪ B) ∖ C