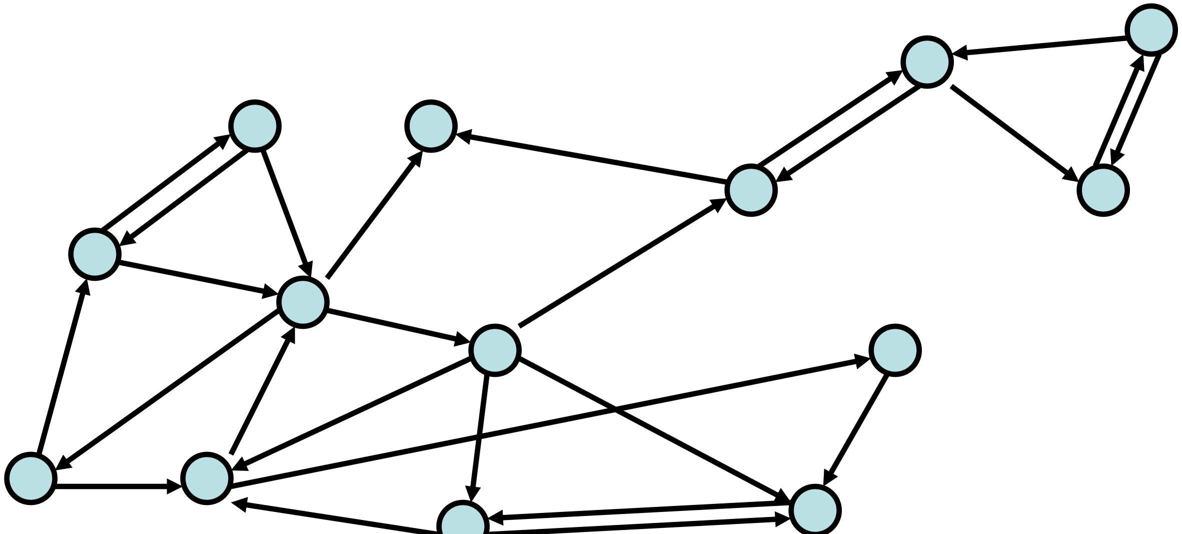
CSE 311: Foundations of Computing

Lecture 27a: Relations and Directed Graphs



Final Exam

Final Exam Practice is up on the website.
We will have three review sessions:

- Thursday from 4:30 – 7:30 in EEB 105
- Saturday from 4:00 – 7:00 in EEB 125
- Sunday from 4:00 – 7:00 in EEB 125

Enjoy!

Epsilon Closure?

One of the major reasons that epsilonClosure was so difficult is that we lacked a way of communicating ideas about the “arrows” in an FSM

Epsilon Closure?

Remember, this course is about the FOUNDATIONS for computing.

We want to give you clean, concise ways of talking about things.

Epsilon Closure?

Last lecture, we talked about functions as a way of discussing infinity.

Now, let's generalize functions.

$$f(x) = y$$

Relations

Let A and B be sets,
A binary relation from A to B is a subset of $A \times B$

Let A be a set,
A binary relation on A is a subset of $A \times A$

Relations You Already Know!

$\geq$ on $\mathbb{N}$

That is: $\{(x,y) : x \geq y \text{ and } x, y \in \mathbb{N}\}$

$<$ on $\mathbb{R}$

That is: $\{(x,y) : x < y \text{ and } x, y \in \mathbb{R}\}$

$=$ on Σ^*

That is: $\{(x,y) : x = y \text{ and } x, y \in \Sigma^*\}$

$\subseteq$ on P(U) for universe U

That is: $\{(A,B) : A \subseteq B \text{ and } A, B \in P(U)\}$

Relation Examples

$R_1 = \{(a, 1), (a, 2), (b, 1), (b, 3), (c, 3)\}$

$R_2 = \{(x, y) \mid x \equiv y \pmod 5 \}$

$R_3 = \{(c_1, c_2) \mid c_1$ is a prerequisite of $c_2 \}$

$R_4 = \{(s, c) \mid$ student s had taken course c $\}$

Perhaps most importantly…

The “transitions” in a DFA/NFA are a relation!

They say “for a particular character, these two states are “related”.

Properties of Relations

Let R be a relation on A.

R is reflexive iff $(a,a) \in R$ for every $a \in A$

R is symmetric iff $(a,b) \in R$ implies $(b, a) \in R$

R is antisymmetric iff $(a,b) \in R$ and $a \neq b$ implies $(b,a) \notin R$

R is transitive iff $(a,b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$

Combining Relations

Let R be a relation from A to B.
Let S be a relation from B to C.

The composition of R and S, $S \circ R$ is the relation from A to C defined by:

$$S \circ R = \{(a, c) \mid \exists\, b \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}$$

Intuitively, a pair is in the composition if there is a "connection" from the first to the second.

Powers of a Relation

Let R be a relation on A.

$R^2 = R \circ R = \{(a, c) : \exists b\ ((a,b) \in R \text{ and } (b, c) \in R$

$R^0 = \{(a, c) : a \ \in A\} = A$

$R^1 = \{(a, b) : (a, b) \in R\} = R$

$R^{n+1} = R^n \circ R$

Epsilon Closure...

The epsilonClosure of the epsilon transitions is R^*

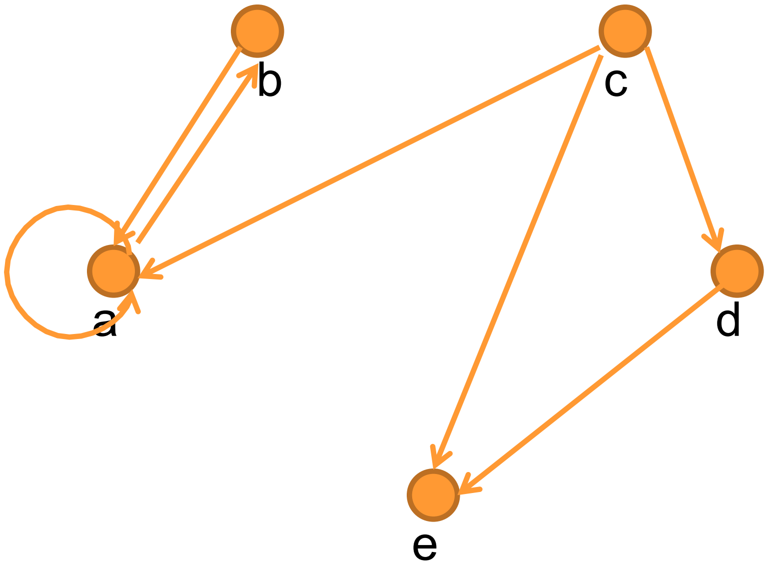
We keep on composing the relation over and over until there's nothing left to add.

This is called the "transitive closure" of a relation.

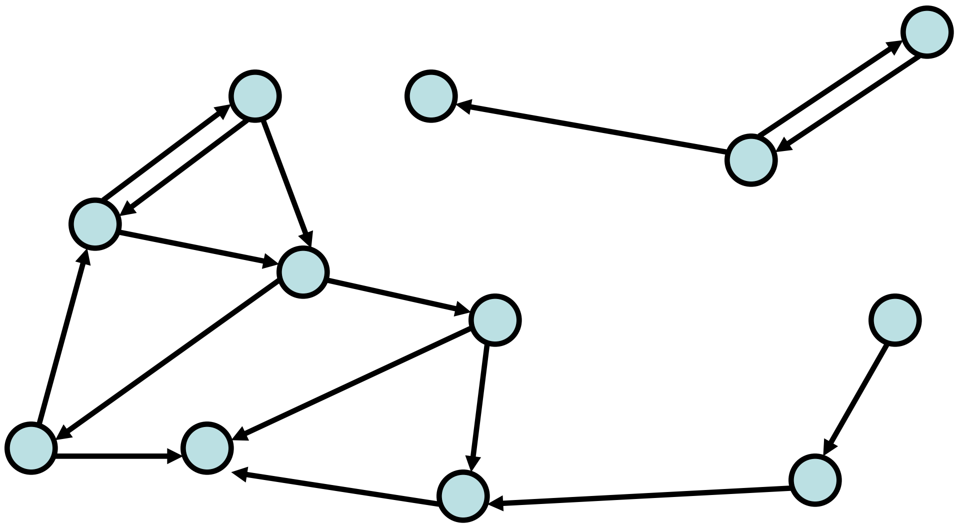
Representation of Relations

Directed Graph Representation (Digraph)

{(a, b), (a, a), (b, a), (c, a), (c, d), (c, e) (d, e) }



Transitive-Reflexive Closure



Add the minimum possible number of edges to make the relation transitive and reflexive.

The transitive-reflexive closure of a relation R is the connectivity relation R*

n-ary Relations

Let $A_1, A_2, \ldots, A_n$ be sets. An **n-ary** relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$.

Relational Databases

STUDENT

Student_Name	ID_Number	Office	GPA
Knuth	328012098	022	4.00
Von Neuman	481080220	555	3.78
Russell	238082388	022	3.85
Einstein	238001920	022	2.11
Newton	1727017	333	3.61
Karp	348882811	022	3.98
Bernoulli	2921938	022	3.21

Relational Databases

STUDENT

Student_Name	ID_Number	Office	GPA	Course
Knuth	328012098	022	4.00	CSE311
Knuth	328012098	022	4.00	CSE351
Von Neuman	481080220	555	3.78	CSE311
Russell	238082388	022	3.85	CSE312
Russell	238082388	022	3.85	CSE344
Russell	238082388	022	3.85	CSE351
Newton	1727017	333	3.61	CSE312
Karp	348882811	022	3.98	CSE311
Karp	348882811	022	3.98	CSE312
Karp	348882811	022	3.98	CSE344
Karp	348882811	022	3.98	CSE351
Bernoulli	2921938	022	3.21	CSE351

What's not so nice?

relational databases

STUDENT

Student_Name	ID_Number	Office	GPA
Knuth	328012098	022	4.00
Von Neuman	481080220	555	3.78
Russell	238082388	022	3.85
Einstein	238001920	022	2.11
Newton	1727017	333	3.61
Karp	348882811	022	3.98
Bernoulli	2921938	022	3.21

TAKES

ID_Number	Course
328012098	CSE311
328012098	CSE351
481080220	CSE311
238082388	CSE312
238082388	CSE344
238082388	CSE351
1727017	CSE312
348882811	CSE311
348882811	CSE312
348882811	CSE344
348882811	CSE351
2921938	CSE351

Better

database operations: projection

Find all offices: $\Pi_{\text{Office}}(\text{STUDENT})$

Office
022
555
333

Find offices and GPAs: $\Pi_{\text{Office,GPA}}(\text{STUDENT})$

Office	GPA
022	4.00
555	3.78
022	3.85
022	2.11
333	3.61
022	3.98
022	3.21

database operations: selection

Find students with GPA > 3.9 : $\sigma_{GPA>3.9}(STUDENT)$

Student_Name	ID_Number	Office	GPA
Knuth	328012098	022	4.00
Karp	348882811	022	3.98

Retrieve the name and GPA for students with GPA > 3.9:
$\Pi_{Student\_Name,GPA}(\sigma_{GPA>3.9}(STUDENT))$

Student_Name	GPA
Knuth	4.00
Karp	3.98

database operations: natural join

Student ⋈ Takes

Student_Name	ID_Number	Office	GPA	Course
Knuth	328012098	022	4.00	CSE311
Knuth	328012098	022	4.00	CSE351
Von Neuman	481080220	555	3.78	CSE311
Russell	238082388	022	3.85	CSE312
Russell	238082388	022	3.85	CSE344
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Karp	348882811	022	3.98	CSE344
Karp	348882811	022	3.98	CSE351
Bernoulli	2921938	022	3.21	CSE351

Examples

(a,b) ∈ Parent iff b is a parent of a

(a,b) ∈ Sister iff b is a sister of a

When is (x,y) ∈ Sister ∘ Parent?

When there is a person z, where (x, z) ∈ Parent and (z, y) ∈ Sister.
That is, z is a parent of x, and y and z are sisters. Or, y is x's Aunt.

When is (x,y) ∈ Parent ∘ Sister?

When there is a person z, where (x, z) ∈ Sister and (z, y) ∈ Parent.
That is, z and x are sisters, and y is a parent of z. Or, y is x's parent.

S ∘ R = {(a, c) | ∃ b such that (a,b) ∈ R and (b,c) ∈ S}

Examples

Using the relations: Parent, Child, Brother, Sister, Sibling, Father, Mother, Husband, Wife express:

Uncle: b is an uncle of a

$(a,b) \in (\text{Brother} \circ \text{Parent} \cup \text{Husband} \circ \text{Sibling} \circ \text{Parent})$

Cousin: b is a cousin of a

$(a,b) \in (\text{Child} \circ \text{Sibling} \circ \text{Parent})$

Relational Composition using Digraphs

If $S = \{(2,2), (2,3), (3,1)\}$ **and** $R = \{(1,2), (2,1), (1,3)\}$

Compute $S \circ R$

(1, 2) ∈ R and (2, 2) ∈ S means (1, 2) ∈ S ∘ R
(1, 2) ∈ R and (2, 3) ∈ S means (1, 3) ∈ S ∘ R
(2, 1) ∈ R and (2, 2) ∈ S means (2, 2) ∈ S ∘ R

Connectivity In Graphs

Two vertices in a graph are connected iff there is a path between them.

Let R be a relation on a set A. The connectivity relation R*
consists of the pairs (a,b) such that there is a path from a to b in R.

$$R^* = \bigcup_{k=0}^{\infty} R^k$$

Note: The text uses the wrong definition of this quantity.
What the text defines (ignoring k=0) is usually called R^+

Properties of Relations (again)

Let R be a relation on A.

R is reflexive iff $(a,a) \in R$ for every $a \in A$

R is symmetric iff $(a,b) \in R$ implies $(b, a) \in R$

R is transitive iff $(a,b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$