CSE 311

Foundations of Computing I

Pre-Lecture Problem

Suppose that p, and $p \rightarrow (q \wedge r)$ are true. Is q true? Can you prove it with equivalences?

$p \wedge p \rightarrow (q \wedge r) \equiv q$

How I 311?

(1) Try Problem ~30 min

(2) Stop! (2.2) ask for help

not true because...

$p \equiv T$, $q \equiv T$, $r \equiv F$

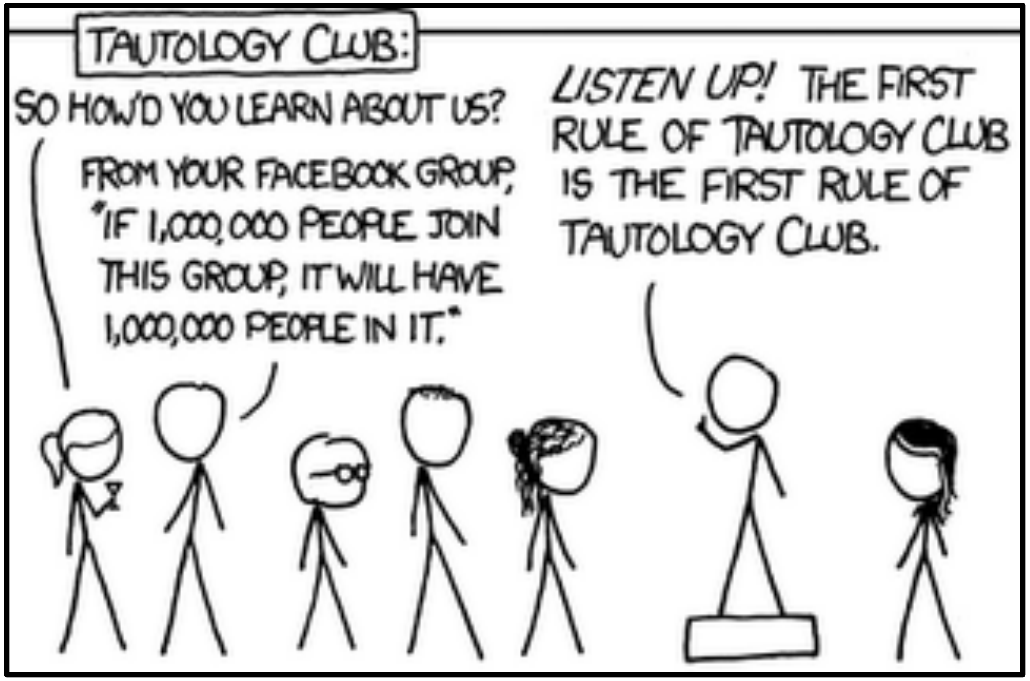
Proof From Last Time

Show that r follows from p, p → q, and q → r

1.	p	Given
2.	p → q	Given
3.	q → r	Given
4.	q	MP: 1, 2
5.	r	MP: 3, 4

CSE 311: Foundations of Computing

Lecture 7: Proofs



Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	Given
2.	$\neg q$	Given
3.	$\neg q \rightarrow \neg p$	Contrapositive: 1
4.	$\neg p$	MP: 2, 3

Inference Rules

If I have a proof of A and
I have a proof of B....

Requirements: $\dfrac{A \qquad B}{}$

Conclusions: $\therefore\ C \quad D$

Then, I have also proven C

Then, I have also proven D

Example (Modus Ponens):

$$\frac{A \qquad A \rightarrow B}{\therefore \qquad B}$$

If I have a proof of A and A → B,
Then I have a proof of B.

Axioms

If I have nothing...

Requirements:

Conclusions: $\therefore$ C D

Then, I have also proven C

Then, I have also proven D

Example (Excluded Middle):

$\therefore A \vee \neg A$

I have a proof of $A \vee \neg A$.

More Inference Rules

Each connective has an "introduction rule" and an "elimination rule"

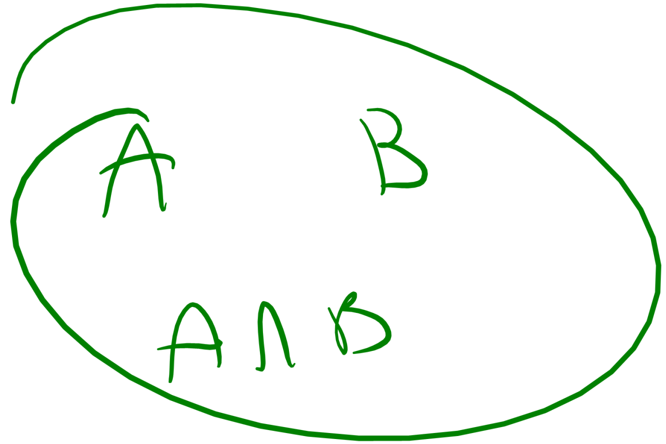
Consider "and". To know A ∧ B is true, what do we need to know…?

A	B	A ∧ B

∧ Introduction

A ∧ B

∴ A B



More Inference Rules

Each connective has an "introduction rule" and an "elimination rule"

Consider "and". To know $A \wedge B$ **is true, what do we need to know…?**

A	B	A ∧ B
T	**T**	**T**
T	F	F
T	T	F
T	F	F

The only case $A \wedge B$ **is true is when A and B are both true.**

∧ Introduction

$$\frac{A \qquad B}{\therefore\ A \wedge B}$$

So, we can only prove $A \wedge B$ **if we already have a proof for A and we already have a proof for B.**

More Inference Rules

Each connective has an "introduction rule" and an "elimination rule"

"Elimination" rules go the other way. If we know A ∧ B**, then what do we know about A and B individually?**

A	B	A ∧ B
T	**T**	**T**
T	F	F
T	T	F
T	F	F

When A ∧ B **is true, then A is true and B is true.**

∧ Elimination

$$\frac{A \land B}{\therefore\ A \qquad B}$$

So, we if we can prove A ∧ B**, then we can also prove A and we can also prove B.**

Proofs

Show that r follows from $p, p \to q$, **and** $(p \wedge q) \to r$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

1.	p	Given
2.	$p \to q$	Given
3.	$(p \wedge q) \to r$	Given
4.	q	MP: 1, 2
5.	$p \wedge q$	$\wedge$ Intro: 1, 4
6.	r	MP: 3, 5

Modus Ponens

$$\frac{A \qquad A \to B}{\therefore\ B}$$

$\wedge$ Introduction

$$\frac{A \qquad B}{\therefore\ A \wedge B}$$

$\wedge$ Elimination

$$\frac{A \wedge B}{\therefore\ A \qquad B}$$

Proofs

Show that r follows from $p, p \to q,$ and $p \land q \to r$

Two visuals of the same proof.
We will focus on the top one,
but if the bottom one helps
you think about it, that's great!

1.	p	Given
2.	$p \to q$	Given
3.	q	MP: 1, 2
4.	$p \land q$	Intro $\land$: 1, 3
5.	$p \land q \to r$	Given
6.	r	MP: 4, 5

$$\dfrac{\dfrac{p \qquad \dfrac{p \qquad p \to q}{q}\,\text{MP}}{p \land q}\,\text{Intro } \land \qquad p \land q \to r}{r}\,\text{MP}$$

Simple Propositional Inference Rules

	Elimination	Introduction
$\wedge$	**$\wedge$ Elimination** $\dfrac{A \wedge B}{\therefore A \qquad B}$	**$\wedge$ Introduction** $\dfrac{A \qquad B}{\therefore A \wedge B}$
$\vee$	**$\vee$ Elimination** $\dfrac{A \vee B \qquad \neg A}{\therefore B}$	**$\vee$ Introduction** $\dfrac{A}{\therefore A \vee B \qquad B \vee A}$
$\rightarrow$	**Modus Ponens** $\dfrac{A \qquad A \rightarrow B}{\therefore B}$	$\dfrac{p \Rightarrow q}{\therefore p \rightarrow q}$ Direct Proof Rule Not like other rules

Important: Application of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \rightarrow q$ Given

2. $(p \vee r) \rightarrow q$ Intro $\vee$: 1

Important: Application of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \rightarrow q$ Given

~~2. $(p \vee r) \rightarrow q$ Intro $\vee$: 1~~

Does not follow! e.g . p=**F**, q=**F**, r=**T**

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

43. q

44. $q \rightarrow \neg r$ Given

45. $\neg r$ MP: 44, 43

Idea: Work backwards!

Proofs

Prove that ¬r follows from p ∧ s, q → ¬r, and ¬s ∨ q.

<u>Idea: Work backwards!</u>

We want to eventually get ¬r. How?

- We can use $q \to \neg r$ to get there.

45. $\neg r$

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

Idea: Work backwards!

We want to eventually get $\neg r$. How?

- We can use $q \rightarrow \neg r$ to get there.
- The justification between 44 and 45 looks like "implication elim" which is MP.

44.	$q \rightarrow \neg r$	Given
45.	$\neg r$	MP: 44, ?

- So, we can justify line 45 now!

Proofs

Prove that ¬r follows from p ∧ s, Used! and ¬s ∨ q.

41. ¬¬s

42. ¬s ∨ q Given

43. q ? ∨ Elim: 42, 41

44. $q \rightarrow \neg r$ **Given**

45. $\neg r$ **MP: 44, 43**

Idea: Work backwards!

We want to eventually get ¬r. How?

- Now, we have a new "hole"
- We need to prove q…
  - Notice that at this point, if we prove q, we've proven $\neg r$…

Proofs

Prove that ¬r follows from p ∧ s, Used! and ¬s ∨ q.

Idea: Work backwards!

We want to eventually get q. How?

- Find a relevant given!

42.	$\neg s \vee q$	Given	
43.	q	?	This looks like or-elimination.
44.	$q \rightarrow \neg r$	Given	
45.	$\neg r$	MP: 44, 43	

Proofs

Prove that $\neg r$ follows from $p \land s$, Used! and Used!

41. $\neg\neg s$ It's more likely that $\neg\neg s$ shows up as s...

42. $\neg s \lor q$ Given

43. q $\lor$ Elim: 42, 41

44. $q \rightarrow \neg r$ Given

45. $\neg r$ MP: 44, 43

Proofs

Prove that $\neg r$ follows from $p \wedge s$, Used! and Used!

39. $p \wedge s$ Given

40. s Use our last given! ∧ Elim: 39

41. $\neg\neg s$ Double Negation: 40

42. $\neg s \vee q$ Given

43. q $\vee$ Elim: 42, 41

44. $q \rightarrow \neg r$ Given

45. $\neg r$ MP: 44, 43

Remember, we're allowed to use equivalences!

Proofs

Prove that $\neg r$ follows from Used! Used! and Used!

We don't have any holes in the proof left! We're done!

39.	$p \wedge s$	Given
40.	s	$\wedge$ Elim: 39
41.	$\neg\neg s$	Double Negation: 40
42.	$\neg s \vee q$	Given
43.	q	$\vee$ Elim: 42, 41
44.	$q \rightarrow \neg r$	Given
45.	$\neg r$	MP: 44, 43

Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

Well, almost, let's renumber the steps:

1.	$p \wedge s$	Given
2.	s	$\wedge$ Elim: 1
3.	$\neg\neg s$	Double Negation: 2
4.	$\neg s \vee q$	Given
5.	q	$\vee$ Elim: 4, 3
6.	$q \rightarrow \neg r$	Given
7.	$\neg r$	MP: 6, 5

To Prove An Implication: $A \to B$

- We use the direct proof rule
- The "pre-requisite" for using the direct proof rule is that we **write a proof** that **Given** A, we can prove B.
- The direct proof rule:

  If you have such a proof then you can conclude that $p \to q$ is true

Example: Prove $p \to (p \vee q)$.

proof subroutine

If p, then $p \vee q$

1.	p	Assumption
2.	$p \vee q$	Intro $\vee$: 1
3.	$p \to (p \vee q)$	Direct Proof Rule

Proofs using the direct proof rule

Show that $p \to r$ follows from q and $(p \wedge q) \to r$

1.	q	Given
2.	(p ∧ q) → r	Given
3.1.	p	Assumption
3.2.	p ∧ q	Intro ∧: 1, 3.1
3.3.	r	MP: 2, 3.2
3.	p → r	Direct Proof Rule

This is a proof of $p \to r$

If we know p is true... Then, we've shown r is true

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

There MUST be an application of the Direct Proof Rule to prove this implication.

Where do we start? We have no givens…

Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

Idea:

Assume $p \wedge q$

⋮ Prove $p \vee q$

$\Sigma, p \wedge q \Rightarrow (p \vee q)$ by DPR

Example

Prove: $(p \land q) \rightarrow (p \lor q)$

1.1.	$p \land q$	Assumption
1.2.	p	Elim $\land$: 1.1
1.3.	$p \lor q$	Intro $\lor$: 1.2
1.	$(p \land q) \rightarrow (p \lor q)$	Direct Proof Rule

E Rules

Both ways

IR & Ax