CSE 311

Foundations of Computing I

Some Reminders/Hints for HW3

Style matters in proofs!

Do NOT manipulate large statements by equivalences! This is horrible style and will lose points if there is a significantly cleaner proof.

Our inference rules can only prove things true. Do not "prove the negation false"! It doesn't make sense.

After you're done writing your proof, you should proof-read (heh) it.

Set Operations

$$A \cup B = \{ x : (x \in A) \vee (x \in B) \}$$ **Union**

$$A \cap B = \{ x : (x \in A) \wedge (x \in B) \}$$ **Intersection**

$$A \setminus B = \{ x : (x \in A) \wedge (x \notin B) \}$$ **Set Difference**

A = {1, 2, 3}
B = {4, 5, 6}
C = {3, 4}

[6] = {1, 2, ..., 6}

QUESTIONS

Using A, B, C and set operations, make…

[6] = $A \cup B$

{3} = $A \cap C$

{1,2} =

Set Operations

$$A \cup B = \{ x : (x \in A) \vee (x \in B) \}$$ Union

$$A \cap B = \{ x : (x \in A) \wedge (x \in B) \}$$ Intersection

$$A \setminus B = \{ x : (x \in A) \wedge (x \notin B) \}$$ Set Difference

A = {1, 2, 3}
B = {4, 5, 6}
C = {3, 4}

<u>QUESTIONS</u>

Using A, B, C and set operations, make…

[6] = $A \cup B = A \cup B \cup C$

{3} = $C \setminus B = A \setminus B = A \cap B$

{1,2} = $A \setminus C = (A \cup B) \setminus C$

More Set Operations

$$A \oplus B = \{\, x : (x \in A) \oplus (x \in B)\}$$

Symmetric Difference

$$\bar{A} = \{\, x : x \notin A\,\}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 4, 2, 6}
C = {1, 2, 3, 4}

<u>QUESTIONS</u>

Let S = {1, 2}.
If the universe is A, then $\bar{S}$ is... {3}
If the universe is B, then $\bar{S}$ is... {4, 6}
If the universe is C, then $\bar{S}$ is...

More Set Operations

$$A \oplus B = \{\, x : (x \in A) \oplus (x \in B)\}$$

Symmetric Difference

$$\bar{A} = \{\, x : x \notin A \,\}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 4, 2, 6}
C = {1, 2, 3, 4}

<u>QUESTIONS</u>

Let S = {1, 2}.

If the universe is A, then $\bar{S}$ is... **A ∖ S = {3}**

If the universe is B, then $\bar{S}$ is... **B ∖ S = {4, 6}**

If the universe is C, then $\bar{S}$ is... **C ∖ S = {3, 4}**

Power Set

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) = \{\, B : B \subseteq A \,\}$$

- Let Days = {M, W, F}. Suppose we wanted to know the possible ways that we could allocate class days to be cancelled. Let's call this set $\mathcal{P}(\text{Days})$.

e.g. $\mathcal{P}(\text{Days}) = \{$

e.g. $\mathcal{P}(\emptyset) = ?$

Power Set

- Power Set of a set A = set of all subsets of A

$$\mathcal{P}(A) = \{ B : B \subseteq A \}$$

- Let Days = {M, W, F}. Suppose we wanted to know the possible ways that we could allocate class days to be cancelled. Let's call this set $\mathcal{P}(\text{Days})$.

e.g. $\mathcal{P}(\varnothing)$ = {Ø}

e.g.
$$\mathcal{P}(\text{Days}) = \{ \varnothing, \{M\}, \{W\}, \{F\}, \{M,W\}, \{W,F\}, \{M,F\}, \{M,W,F\} \}$$

Cartesian Product

$$A \times B = \{\ (a, b) : a \in A, b \in B\ \}$$

$\mathbb{R} \times \mathbb{R}$ **is the real plane. You've seen ordered pairs before.**

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ **is "the set of all pairs of integers"**

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}.

A $\times \emptyset = \{(a, b) : a \in A \ \wedge\ b \in \emptyset\} = \{(a, b) : a \in A \ \wedge\ \text{F}\} = \emptyset$

Russell's Paradox

$$S = \{ x : x \notin x \}$$

Suppose for contradiction that S ∈ S...

So, $S \notin S$.

Suppose for contradiction $S \notin S$.

So, $S \in S$

$\{1, 2, 3\} \in \{1, 2, 3\}$

$\{1, 2, 3\} \subseteq \{1, 2, 3\}$

Russell's Paradox

$$S = \{ x : x \notin x \}$$

Suppose for contradiction that S ∈ S. Then, by definition of S, S ∉ S, but that's a contradiction.

Suppose for contradiction that S ∉ S. Then, by definition of the set comprehension, S ∈ S, but that's a contradiction.

This is reminiscent of the truth value of the statement "This statement is false."

It's Boolean algebra again

- Definition for ∪ based on ∨

$$A \cup B = \{ x : (x \in A) \vee (x \in B) \}$$

- Definition for ∩ based on ∧

$$A \cap B = \{ x : (x \in A) \wedge (x \in B) \}$$

- Complement works like ¬

$$\overline{A} = \{ x : x \notin A \}$$

(with respect to universe U)

De Morgan's Laws

Prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Let U be the universe.

$$\overline{A \cup B} = \{x : x \in \overline{A \cup B}\}$$

$$= \{x : x \notin A \cup B\}$$

$$= \{x : \neg(x \in A \cup B)\}$$

$$= \{x : \neg(x \in A \vee x \in B)\}$$

$$= \{x : x \notin A \wedge x \notin B\}$$

$$= \{x : x \in \overline{A} \wedge x \in \overline{B}\}$$

$$= \{x : x \in \overline{A} \cap \overline{B}\} = \overline{A} \cap \overline{B}$$

De Morgan's Laws

Prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Let U be the universe.

$$
\begin{aligned}
\overline{A \cup B} &= \{x : x \notin A \cup B\} \\
&= \{x : \neg(x \in A \cup B)\} \\
&= \{x : \neg\big((x \in A) \vee (x \in B)\big)\} \\
&= \{x : (x \notin A) \wedge (x \notin B)\} \\
&= \{x : \big(x \in \overline{A}\big) \wedge \big(x \in \overline{B}\big)\} \\
&= \{x : \big(x \in \overline{A}\big)\} \cap \{x : \big(x \in \overline{B}\big)\} \\
&= \overline{A} \cap \overline{B}
\end{aligned}
$$

→ **Prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$**

De Morgan's Laws

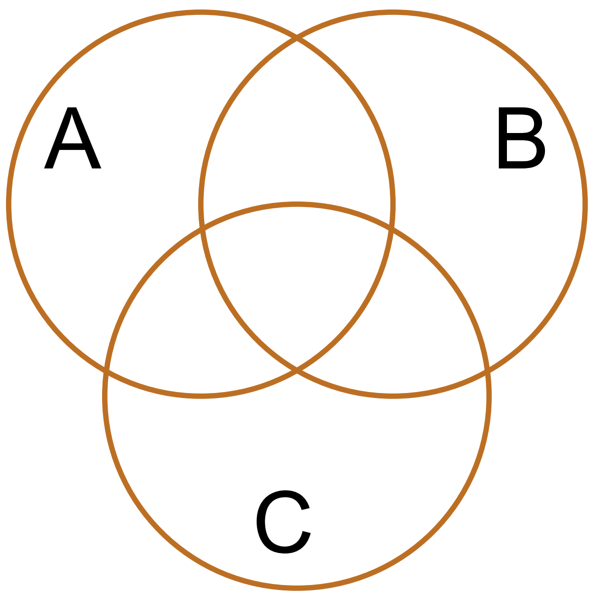
Prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$

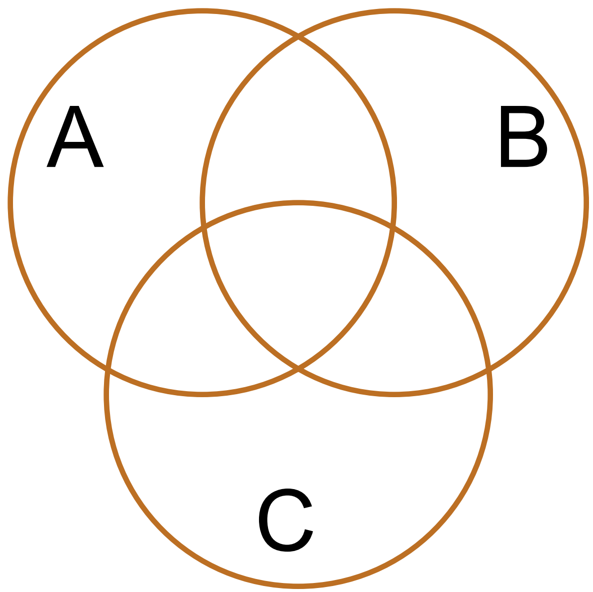
Let U be the universe.

$$\begin{aligned}
\overline{A \cap B} &= \{x : x \notin A \cap B\} \\
&= \{x : \neg(x \in A \cap B)\} \\
&= \{x : \neg\big((x \in A) \wedge (x \in B)\big)\} \\
&= \{x : (x \notin A) \vee (x \notin B)\} \\
&= \{x : \big(x \in \overline{A}\big) \vee \big(x \in \overline{B}\big)\} \\
&= \{x : \big(x \in \overline{A}\big)\} \cup \{x : \big(x \in \overline{B}\big)\} \\
&= \overline{A} \cup \overline{B}
\end{aligned}$$

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$





One More

Prove $(A \cap B) \subseteq A$

Remember the definition of subset?

$$X \subseteq Y \leftrightarrow \forall x\,(x \in X \rightarrow x \in Y)$$

Let x be arbitrary. Suppose $x \in A \cap B$.

So, by def. of $A \cap B$, $x \in A$ and $x \in B$.

So, $x \in A$.

So, $(A \cap B) \subseteq A$

One More

Prove $(\mathrm{A} \cap B) \subseteq A$

Remember the definition of subset?

$$X \subseteq Y \leftrightarrow \forall x\,(x \in X \rightarrow x \in Y)$$

Let x **be an arbitrary element of** $\mathrm{A} \cap B$**. Then, by definition of** $\mathrm{A} \cap B$**,** $\mathrm{x} \in A$ **and** $\mathrm{x} \in B$**. It follows that** $\mathrm{x} \in A$**, as required.**

Representing Sets Using Bits

- Suppose universe U is $\{1,2,\dots,n\}$
- Can represent set $B \subseteq U$ as a vector of bits:

  $b_1 b_2 \dots b_n$ where $b_i = 1$ when $i \in B$

  $b_i = 0$ when $i \notin B$

  – Called the *characteristic vector* of set B

- Given characteristic vectors for A and B
  – What is characteristic vector for $A \cup B$? $A \cap B$?

UNIX/Linux File Permissions

- `ls -l`

```
drwxr-xr-x ... Documents/
-rw-r--r-- ... file1
```

- Permissions maintained as bit vectors
  - Letter means bit is 1
  - “–” means bit is 0.

CSE 311: Foundations of Computing

Lecture 10: Modular Arithmetic

Number Theory (and applications to computing)

- Branch of Mathematics with direct relevance to computing
- Many significant applications
  - Cryptography
  - Hashing
  - Security
- Important tool set

Modular Arithmetic

- Arithmetic over a finite domain

- In computing, almost all computations are over a finite domain

$(100 + 10) \% m$

I’m ALIVE!

```
public class Test {
    final static int SEC_IN_YEAR = 364*24*60*60*100;
    public static void main(String args[]) {
        System.out.println(
            “I will be alive for at least ” +
            SEC_IN_YEAR * 101 + “ seconds.”
        );
    }
}
```

I’m ALIVE!

```
public class Test {
    final static int SEC_IN_YEAR = 364*24*60*60*100;
    public static void main(String args[]) {
        System.out.println(
            “I will be alive for at least ” +
            SEC_IN_YEAR * 101 + “ seconds.”
        );
    }
}
```

2^{32}

2^{64}

```
 ----jGRASP exec: java Test
I will be alive for at least -186619904 seconds.

 ----jGRASP: operation complete.
```

Divisibility

Definition: "a divides b"

For $a \in \mathbb{Z}, b \in \mathbb{Z}$ with $a \neq 0$:

$$a \mid b \leftrightarrow \exists(k \in \mathbb{Z})\ \mathrm{b} = \mathrm{ka}$$

Check Your Understanding. Which of the following are true?

5 | 1	**25 | 5**	**5 | 5**	**3 |2**
1 | 5	**5 | 25**	**0 | 1**	**2 | 3**

$1 = k \cdot 5$

$5 = 1 \cdot k$

$a \mid b \leftrightarrow \frac{b}{a} \in \mathbb{Z}$

Divisibility

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$$a \mid b \leftrightarrow \exists(k \in \mathbb{Z})\ \mathrm{b} = \mathrm{ka}$$

Check Your Understanding. Which of the following are true?

5 | 1	**25 | 5**	**5 | 5**	**3 |2**
5 | 1 iff 1 = 5k	25 | 1 iff 1 = 25k	5 | 5 iff 5 = 5k	3 | 2 iff 2 = 3k
1 | 5	**5 | 25**	**0 | 1**	**2 | 3**
1 | 5 iff 5 = 1k	1 | 25 iff 25 = 1k	0 | 1 iff 1 = 0k	2 | 3 iff 3 = 2k

Division Theorem

Division Theorem

For $a \in \mathbb{Z}$, $d \in \mathbb{Z}^{+}$:
Then, there exists *unique* integers q, r with $0 \leq r < d$
such that $a = dq + r$.

To put it another way, if we take a/d, we get a dividend and a remainder: $q = a$ **div** d $r = a$ **mod** d

$\frac{a}{d}$

$-1 \% 5 =$

Note: $r \geq 0$ even if $a < 0$.
Not quite the same as `a % d`.

Division Theorem

Division Theorem

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Then, there exists *unique* integers q, r with $0 \leq r < d$
such that $a = dq + r$.

To put it another way, if we take a/d, we get a dividend and a remainder: $q = a$ **div** d $\quad r = a$ **mod** d

```
public class Test2 {
    public static void main(String args[]) {
        int a = -5;
        int d = 2;
        System.out.println(a % d);
    }
}
```

```
----jGRASP exec: java Test2
-1

----jGRASP: operation complete.
```

Note: $r \geq 0$ even if $a < 0$.
Not quite the same as `a % d`.

Arithmetic, mod 7

$5 +_7 6 = 11 \bmod 7 = 4$

$$a +_7 b = (a + b) \bmod 7$$

$$a \times_7 b = (a \times b) \bmod 7$$

$5 \times_7 6 = 30 \bmod 7 = 2$

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Modular Arithmetic

Definition: "a is congruent to b modulo m"

For $a \in \mathbb{Z}, b \in \mathbb{Z}, \mathrm{m} \in \mathbb{Z}$:

$$(a \equiv b \pmod{m}) \leftrightarrow m \mid (a - b)$$

$km = a - b$

Check Your Understanding. What do each of these mean? When are they true?

A ≡ 0 (mod 2)

1 ≡ 0 (mod 4)

A ≡ -1 (mod 17)

Modular Arithmetic

Definition: "a is congruent to b modulo m"

For $a \in \mathbb{Z}, b \in \mathbb{Z}, \mathrm{m} \in \mathbb{Z}$:

$$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$$

Check Your Understanding. What do each of these mean? When are they true?

A ≡ 0 (mod 2)

This statement is the same as saying "A is even"; so, any A that is even (including negative even numbers) will work.

1 ≡ 0 (mod 4)

This statement is false. If we take it mod 1 instead, then the statement is true.

A ≡ -1 (mod 17)

If A = 17x – 1 = 17x + 16, then it works.

Note that (m – 1) mod m = ((m mod m) + (-1 mod m)) mod m
= (0 + -1) mod m = -1 mod m

Modular Arithmetic: A Property

Let a and b be integers, and let m be a positive integer.
Then, a ≡ b (mod m) if and only if a mod m = b mod m.

Suppose that a ≡ b (mod m).

Suppose that a mod m = b mod m.

Modular Arithmetic: A Property

Let a and b be integers, and let m be a positive integer.
Then, a ≡ b (mod m) if and only if a mod m = b mod m.

Suppose that a ≡ b (mod m).

Then, m | (a – b) by definition of congruence.

So, a – b = km for some integer k by definition of divides.

Therefore, a = b+km.

Taking both sides modulo m we get:

a mod m=(b+km) mod m = b mod m.

Suppose that a mod m = b mod m.

By the division theorem, a = mq + (a mod m) and
b = ms + (b mod m) for some integers q,s.

Then, a – b = (mq + (a mod m)) – (mr + (b mod m))
= m(q – r) + (a mod m – b mod m)
= m(q – r) since a mod m = b mod m

Therefore, m | (a-b) and so $a \equiv b$ (mod m).