Adam Blank Spring 2016

CSE 311

Foundations of Computing I

* All slides are a combined effort between previous instructors of the course

Administrivia

If you want to use a token on HW1-HW3, you need to sign up for it by 11:30pm tonight.

Midterm practice materials are up on the website.

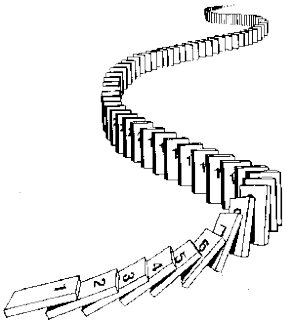
The midterm will be on Wed, May 4 from 4:30pm – 6:00pm in JHN 102

If you cannot make this time, I need to know by **Friday** to schedule a make-up exam.

There will be two review sessions time/location TBD.

CSE 311: Foundations of Computing

Lecture 14: Induction



Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- **Particularly useful for reasoning about programs!**

```
for(int i=0; i < n; n++) { … }
```

- Show P(i) holds after i times through the loop

```
public int f(int x) {
    if (x == 0) { return 0; }
    else { return f(x - 1); }
}
```

- f(x) = x for all values of x ≥ 0 naturally shown by induction.

Prove $\forall(a,b \in \mathbb{Z})\ \forall(n \in \mathbb{N})\ (a \equiv b \pmod n) \rightarrow a^i \equiv b^i \pmod n$

Let $a, b \in \mathbb{Z}$ be arbitrary. Let $i \in \mathbb{N}$ be arbitrary. Suppose $a \equiv b \pmod n$.

We know $(a \equiv b \pmod n \wedge a \equiv b \pmod n) \rightarrow a^2 \equiv b^2 \pmod n$ by multiplying congruences. So, applying this repeatedly, we have:

$$(a \equiv b \pmod n \wedge a \equiv b \pmod n) \rightarrow a^2 \equiv b^2 \pmod n$$
$$(a^2 \equiv b^2 \pmod n \wedge a \equiv b \pmod n) \rightarrow a^3 \equiv b^3 \pmod n$$
$$\dots$$
$$(a^{i-1} \equiv b^{i-1} \pmod n \wedge a \equiv b \pmod n) \rightarrow a^i \equiv b^i \pmod n$$

The "…"s is a problem! We don't have a proof rule that allows us to say "do this over and over".

So, make one!

Domain: Natural Numbers

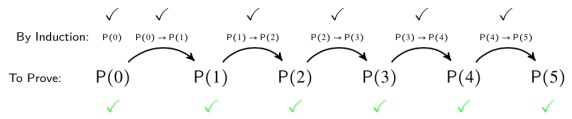
$$\frac{P(0) \qquad \forall\, k\,(P(k) \rightarrow P(k+1))}{\therefore\ \forall\, n\, P(n)}$$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall\, k\,(P(k) \rightarrow P(k+1))$$
$$\therefore\ \forall\, n\, P(n)$$

How does this technique prove P(5)?



First, we prove P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Translating to an English Proof

$$P(0)$$
$$\forall\, k\,(P(k) \rightarrow P(k+1))$$
$$\therefore\ \forall\, n\, P(n)$$

1. Prove P(0) — **Base Case**
2. Let k be an arbitrary integer ≥ 0 — **Inductive Hypothesis**
   3.1. Assume that P(k) is true
   3.2. …
   3.3. Prove P(k+1) is true — **Inductive Step**
3. P(k) → P(k+1) — Direct Proof Rule
4. ∀k (P(k) → P(k+1)) — Intro ∀: 2, 3
5. ∀n P(n) — Induction: 1, 4

Conclusion

Translating To An English Proof

1. Prove P(0) — **Base Case**
2. Let k be an arbitrary integer ≥ 0 — **Inductive Hypothesis**
   3.1. Assume that P(k) is true
   3.2. …
   3.3. Prove P(k+1) is true — **Inductive Step**
3. P(k) → P(k+1) — Direct Proof Rule
4. ∀k (P(k) → P(k+1)) — Intro ∀: 2, 3
5. ∀n P(n) — Induction: 1, 4

Conclusion

Induction Proof Template

[…Define P(n)…]
We will show that P(n) is true for every $n \in \mathbb{N}$ by Induction.
Base Case: *[…proof of P(0) here…]*
Induction Hypothesis:
Suppose P(k) is true for some $k \in \mathbb{N}$.
Induction Step:
We want to prove that P(k+1) is true.
[…proof of P(k+1) here…]
*The proof of P(k+1) **must** invoke the IH somewhere.*
So, the claim is true by induction.

5 Steps To Inductive Proofs In English

Proof:

1. **"We will show that P(n) is true for every n ≥ 0 by Induction."**
2. **"Base Case:" Prove P(0)**
3. **"Inductive Hypothesis:"**
   Assume P(k) is true for some arbitrary integer k ≥ 0"
4. **"Inductive Step:" Want to prove that P(k+1) is true:**
   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it. (Don't assume P(k+1) !!)
5. **"Conclusion: Result follows by induction"**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

- **We could try proving it with properties of summations?**
- **We could use calculus?**
- **Could this be induction?**

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1}$". We go by induction on n.

Base Case (n=0):

Note that $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$, which is exactly $P(0)$.

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$". We go by induction on n.

Base Case (n=0):

Note that $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$, which is exactly $P(0)$.

Induction Hypothesis:

Suppose P(k) is true for some $k \in \mathbb{N}$.

Induction Step:

We want to show P($k+1$). That is, we want to show:

$$\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$$

$\sum_{i=0}^{k+1} 2^i =$ … $=$ … $=$ … $= 2^{n+1} - 1.$ — One of these steps must use the IH.

So, the claim is true for all natural numbers by induction.

Prove 1 + 2 + 4 + … + $2^n = 2^{n+1} - 1$

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$". We go by induction on n.

Base Case (n=0): Note that $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$, which is exactly $P(0)$.

Induction Hypothesis: Suppose P(k) is true for some k ∈ ℕ.

Induction Step: We want to show P($k+1$). That is, we want to show: $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$

Note that $\sum_{i=0}^{k+1} 2^i = \left(\sum_{i=0}^{k} 2^i\right) + 2^{k+1}$ [Splitting the summation]

$= (2^{k+1} - 1) + 2^{k+1}$ [By IH]

$= (2^{k+1} + 2^{k+1}) - 1$ [Assoc. of +]

$= (2(2^{k+1})) - 1$ [Factoring]

$= 2^{k+2} - 1$ [Simplifying]

This is exactly P($k+1$). So, P(k) → P($k+1$).

So, the claim is true for all natural numbers by induction.

Don't bother justifying the "obvious" steps. But make sure you say "by IH" somewhere.

We know (by IH)… $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

We're trying to get… $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1$

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

Prove 1 + 2 + 3 + … + n = n(n+1)/2

Let $P(n)$ be "$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$". We go by induction on n.

Base Case (n=0): Note that $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$, which is exactly $P(0)$.

Induction Hypothesis: Suppose P(k) is true for some k ∈ ℕ.

Induction Step: We want to show P($k+1$). That is, we want to show: $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$

Note that $\sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i\right) + (k+1)$ [Splitting the summation]

$= \left(\frac{k(k+1)}{2}\right) + (k+1)$ [By IH]

$= (k+1)\left(\frac{k}{2} + 1\right) = (k+1)\left(\frac{k+2}{2}\right)$ [Algebra]

$= \frac{(k+1)(k+2)}{2}$ [Algebra]

This is exactly P($k+1$). So, P(k) → P($k+1$).

So, the claim is true for all natural numbers by induction.

We know (by IH)… $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$

We're trying to get… $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$

Our goal is to find a sub-expression of the left that looks like the left side of the IH.

Prove 3 | $2^{2n} - 1$ for all n ≥ 0.

Let $P(n)$ be "3 | $2^{2n} - 1$." We go by induction on n.

Base Case (n=0):

Induction Hypothesis:

Induction Step:

We know (by IH)…

…which means…

We're trying to get…

…which is true if…