CSE 311: Foundations of Computing

Lecture 28: Undecidability and Reductions

Review: Countability vs Uncountability

- To prove a set A countable you must show
  - There exists a listing $x_1, x_2, x_3, \ldots$ such that every element of A is in the list.

- To prove a set B uncountable you must show
  - For *every* listing $x_1, x_2, x_3, \ldots$ there exists some element in B that is not in the list.

  - The diagonalization proof shows how to describe a missing element d in B based on the listing $x_1, x_2, x_3, \ldots$.
    Important: the proof produces a d no matter what the listing is.

Last time: Undecidability of the Halting Problem

CODE(**P**) means "the code of the program **P**"

The Halting Problem

Given: - CODE(**P**) for any program **P**
- input **x**

Output: **true** if **P** halts on input **x**
false if **P** does not halt on input **x**

Theorem [Turing]: **There is no program that solves the Halting Problem**

We use the term "undecidable" since it is a true/false question that is not computable.

Why, intuitively, is this hard?

This really is a legal C program…

What does this program do?

```
_(__,___,____){___/__<=1?_(__,___+1,___
_):!(___%__)?_(__,___+1,0):___%__==___ /
__&&!____?(printf("%d\t",___/__),_(__,_
__+1,0)):___%__>1&&___%__<___/__?_(
__,1+
___,____+!(___/__%(___%__))):___<__*__
?_(__,___+1,____):0;}main(){_(100,0,0);}
```

A “Simple” Program

```
public static void collatz(n) {
    if (n == 1) {
        return 1;
    }
    if (n % 2 == 0) {
        return collatz(n/2)
    }
    else {
        return collatz(3*n + 1)
    }
}
```

11
34
17
52
26
13
40
20
10
5
16
8
4
2
1

What does this program do?

… **on n=11?**

… **on n=10000000000000000001?**

A "Simple" Program

```
public static void collatz(n) {
    if (n == 1) {
        return 1;
    }
    if (n % 2 == 0) {
        return collatz(n/2)
    }
    else {
        return collatz(3*n + 1)
    }
}
```

Nobody knows whether or not this program halts on all inputs!

Trying to solve this has been called a "mathematical disease".

What does this program do?

… **on n=11?**

… **on n=10000000000000000001?**

Last time: Undecidability of the Halting Problem

`CODE(P)` means "the code of the program **P**"

> **The Halting Problem**
>
> **Given:** - CODE(**P**) for any program **P**
> - input **x**
>
> **Output:** **true** if **P** halts on input **x**
> **false** if **P** does not halt on input **x**

Theorem [Turing]: **There is no program that solves the Halting Problem**

Proof: By contradiction.

Assume that a program **H** solving the Halting program does exist. Then program **D** must exist

Does **D**(CODE(**D**)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;        /*    halt    */
    }
}
```

H solves the halting problem implies that

H(CODE(**D**),x) is **true** iff **D**(x) halts, **H**(CODE(**D**),x) is **false** iff not

Note: Even though the program **D** has a **`while`**(true), that doesn't mean that the program **D** actually goes into an infinite loop on input **x**, which is what **H** has to determine

Does D(CODE(D)) halt?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;        /*    halt    */
    }
}
```

H solves the halting problem implies that
H(CODE(D),x) is **true** iff D(x) halts, H(CODE(D)... ot

Suppose that D(CODE(D)) **halts**.
Then, by definition of H it mus...
H(CODE(D), CO...
Which by the defi... D(CODE(D)) **doesn't halt**

Suppose th... **doesn't halt**.
Th... of H it must be that
...ODE(D), CODE(D)) is **false**
Whi... by the definition of D means D(CODE(D)) **halts**

Contradiction!

The ONLY assumption was the program H exists so that assumption must have been false.

Where did the idea for creating D come from?

Connection to diagonalization

Write <P> for CODE(P)

Some possible inputs **x**

$<P_1>$ $<P_2>$ $<P_3>$ $<P_4>$ $<P_5>$ $<P_6>$ ....

All programs **P**

P_1
P_2
P_3
P_4
P_5
P_6
P_7
P_8
P_9
.
.

This listing of all programs really does exist since the set of all Java programs is countable

The goal of this "diagonal" argument is not to show that the listing is incomplete but rather to show that a "flipped" diagonal element is not in the listing

Connection to diagonalization

Write $<P>$ for CODE(P)

Some possible inputs **x**

All programs **P**

	$<P_1>$	$<P_2>$	$<P_3>$	$<P_4>$	$<P_5>$	$<P_6>$	....					
P_1	0	1	1	0	1	1	1	0	0	0	1	...
P_2	1	1	0	1	0	1	1	0	1	1	1	...
P_3	1	0	1	0	0	0	0	0	0	0	1	...
P_4	0	1	1	0	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1	0	1	0	...
P_9	. . . .											
.	. . . .											
.												

(P,x) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

Connection to diagonalization

Write <**P**> for **CODE(P)**

Some possible inputs **x**

All programs **P**

	$\langle P_1\rangle$	$\langle P_2\rangle$	$\langle P_3\rangle$	$\langle P_4\rangle$	$\langle P_5\rangle$	$\langle P_6\rangle$	....					
P_1	0^{1}	1	1	0	1							
P_2	1	1^{0}	0	1	0							
P_3	1	0	1^{0}	0	0							
P_4	0	1	1	0^{1}	1	0	1	1	0	1	0	...
P_5	0	1	1	1	1^{0}	1	1	0	0	0	1	...
P_6	1	1	0	0	0	1^{0}	1	0	1	1	1	...
P_7	1	0	1	1	0	0	0^{1}	0	0	0	1	...
P_8	0	1	1	1	1	0	1	1^{0}	0	1	0	...
P_9	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.
.												

Want behavior of program ***D*** to be like the flipped diagonal, so it can't be in the list of all programs.

(P,x) entry is **1** if program **P** halts on input **x** and **0** if it runs forever

Where did the idea for creating D come from?

```
public static void D(x) {
    if (H(x,x) == true) {
        while (true); /* don't halt */
    }
    else {
        return;        /*    halt    */
    }
}
```

D halts on input code(P) iff H(code(P),code(P)) doesn't halt
iff P doesn't halt on input code(P)

Therefore for any program P, D differs from P on input code(P)

The Halting Problem isn't the only hard problem

- Can use the fact that the Halting Problem is undecidable to show that other problems are undecidable

General method:

Prove that if there were a program deciding B then there would be a way to build a program deciding the Halting Problem.

"B decidable $\rightarrow$ Halting Problem decidable"

Contrapositive:

"Halting Problem undecidable $\rightarrow$ B undecidable"

Therefore B is undecidable

A CSE 141 assignment

Students should write a Java program that:

- Prints "Hello" to the console
- Eventually exits

GradeIt, PracticeIt, etc. need to grade the students.

How do we write that grading program?

WE CAN'T: THIS IS IMPOSSIBLE!

A related undecidable problem

- **HelloWorldTesting Problem:**
  - **Input: CODE(Q)** and **x**
  - **Output:**

    True if **Q** outputs “HELLO WORLD” on input **x**

    False if **Q** does not output “HELLO WORLD” on input **x**

- **Theorem: The HelloWorldTesting Problem is undecidable.**
- **Proof idea: Show that if there is a program T to decide HelloWorldTesting then there is a program H to decide the Halting Problem for code(P) and x.**

A related undecidable problem

- **Suppose there is a program T that solves the HelloWorldTesting problem. Define program H that takes input CODE(P) and x and does the following:**
  - **Creates CODE(Q) from CODE(P) by**

    (1) removing all output statements from CODE(P), and

    (2) adding a System.out.println("HELLO WORLD") **immediately before any spot where P could halt**

    Then runs T on input CODE(Q) and x.

- If **P halts on input x** then **CODE(Q)** prints HELLO WORLD and halts and so **H** outputs **true** (because **T** outputs true on input **CODE(Q)**)
- If **P doesn't halt on input x** then **CODE(Q)** won't print anything since we removed any other print statement from CODE(**Q**) so **H** outputs **false**

We know that such an H cannot exist. Therefore T cannot exist.

The HaltsNoInput Problem

- **Input: CODE(R)** for program **R**
- **Output: True** if **R** halts without reading input

  False otherwise.

Theorem: HaltsNoInput is undecidable

General idea "hard-coding the input":

- Show how to use **CODE(P)** and **x** to build **CODE(R)** so **P halts on input x $\Leftrightarrow$ R halts without reading input**

The HaltsNoInput Problem

"Hard-coding the input":

- Show how to use **CODE(P)** and **x** to build **CODE(R)** so
  P halts on input x ⇔ R halts without reading input

- Replace input statement in **CODE(P)** that reads input **x** into variable **var,** by a hard-coded assignment statement:

  var = x

  to produce **CODE(R)**.

- So if we have a program **N** to decide **HaltsNoInput** then we can use it as a subroutine as follows to decide the Halting Problem, which we know is impossible:
  - On input CODE(**P**) and **x**, produce CODE(**R**). Then run **N** on input CODE(**Q**) and output the answer that **N** gives.

- The impossibility of writing the CSE 141 grading program follows by combining the ideas from the undecidability of HaltsNoInput and HelloWorld.

More Reductions

- Can use undecidability of these problems to show that other problems are undecidable.

- For instance:

$\mathrm{EQUIV}(P, Q)$:	**True**	if $P(x)$ and $Q(x)$ have the same behavior for every input x
	False	otherwise

Rice's theorem

Not *every* problem on programs is undecidable!

Which of these is decidable?

- Input `CODE(P)` and x
  Output: true if P prints "ERROR" on input x
  after less than 100 steps
  false otherwise

- Input `CODE(P)` and x
  Output: true if P prints "ERROR" on input x
  after more than 100 steps
  false otherwise

Rice's Theorem (a.k.a. Compilers Suck Theorem - informal):

Any "non-trivial" property of the input-output behavior of Java programs is undecidable.