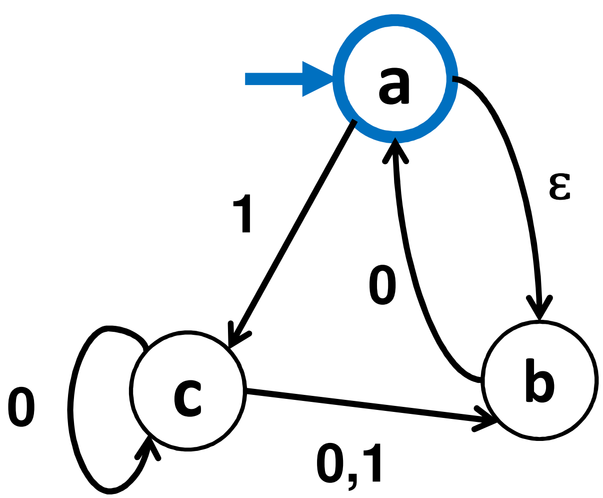
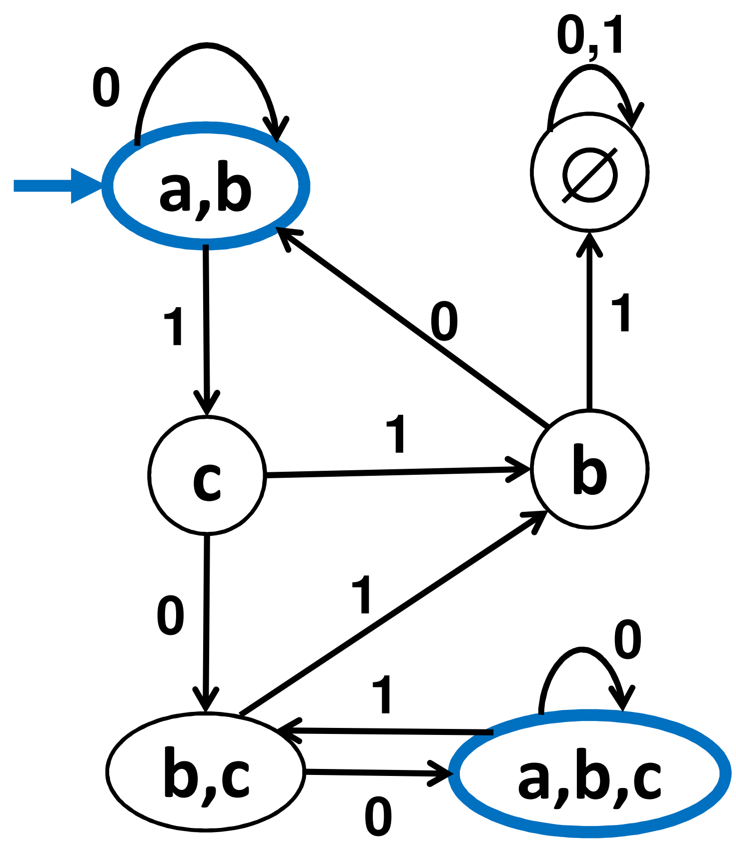
CSE 311: Foundations of Computing

Lecture 25: Pattern Matching, DFA≡NFA≡Regex
Languages vs Representations

Last time: NFA to DFA



NFA



DFA

Exponential Blow-up in Simulating Nondeterminism

- In general the DFA might need a state for every subset of states of the NFA
  - Power set of the set of states of the NFA
  - n-state NFA yields DFA with at most 2^n states
  - We saw an example where roughly 2^n is necessary

    "Is the n^{th} char from the end a 1?"

The famous "P=NP?" question asks whether a similar blow-up is always necessary to get rid of nondeterminism for polynomial-time algorithms

Pattern matching

- Given
  - a string s of n characters
  - a pattern p of m characters
  - usually $m \ll n$
- Find
  - all occurrences of the pattern p in the string s

- Obvious algorithm:
  - try to see if p matches at each of the positions in s

    stop at a failed match and try matching at the next position

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

pattern **p** = x y x y y x y x y x x

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y y x y x y x x

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y

x y x y y x y x y x x

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y

x

x y x y y x y x y x x

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y

x

x y

x y x y y x y x y x x

```
string s = x y x x y x y x y y x y x y x y y x y x y x x
           x y x y
             x
               x y
                 x y x y y
                   x y x y y x y x y x x
```

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y

x

x y

x y x y y

x

x y x y y x y x y x x

```
string s = x y x x y x y x y y x y x y x y y x y x y x x
           x y x y
             x
               x y
                 x y x y y
                   x
                     x y x y y x y x y x x
                       x y x y y x y x y x x
```

```
string s = x y x x y x y x y y x y x y x y y x y x y x x
           x y x y
             x
               x y
                 x y x y y
                   x
                     x y x y y x y x y x x
                       x
                         x y x y y x y x y x x
```

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y

x

x y

x y x y y

x

x y x y y x y x y x x

x

x y x

x y x y y x y x y x x

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y

x

x y

x y x y y

x

x y x y y x y x y x x

x

x y x

x

x y x y y x y x y x x

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y

x

x y

x y x y y

x

x y x y y x y x y x x

x

x y x

x

x

x y x y y x y x y x x

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y

x

x y

x y x y y

x

x y x y y x y x y x x

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x y x

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x

x y x y y

x y x y y x y x y x x

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y

x

x y

x y x y y

x

x y x y y x y x y x x

x

x y x

x

x

x y x y y

x

x y x y y x y x y x x

Worst-case time

$O(mn)$

string **s** = x y x x y x y x y y x y x y x y y x y x y x x

x y x y

x

x y

x y x y y

x

x y x y y x y x y x x

x

x y x

x

x

x y x y y

x

x y x y y x y x y x x

Lots of wasted work

Better pattern matching via finite automata

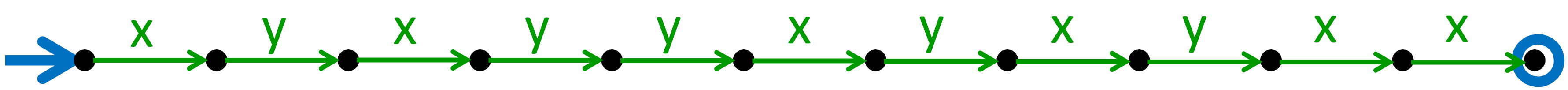
- Build a DFA for the pattern (preprocessing) of size $\boldsymbol{O(m)}$
  - Keep track of the 'longest match currently active'
  - The DFA will have only $\boldsymbol{m} + 1$ states

- Run the DFA on the string $\boldsymbol{n}$ steps

- Obvious construction method for DFA will be $\boldsymbol{O(m^2)}$ but can be done in $\boldsymbol{O(m)}$ time.
- Total $\boldsymbol{O(m+n)}$ time

Building a DFA for the pattern

pattern **p**=x y x y y x y x y x x



Building a DFA for the pattern

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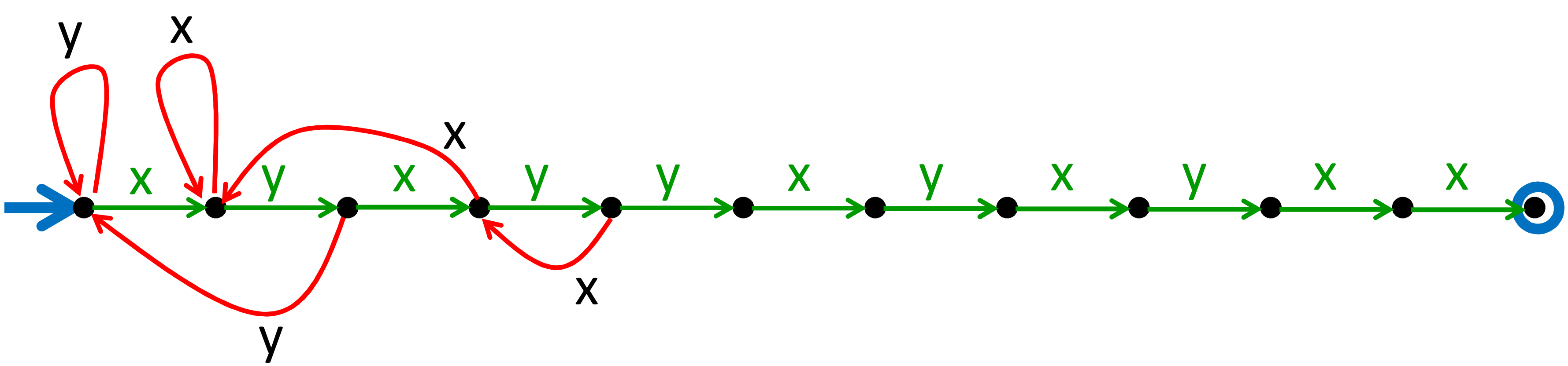

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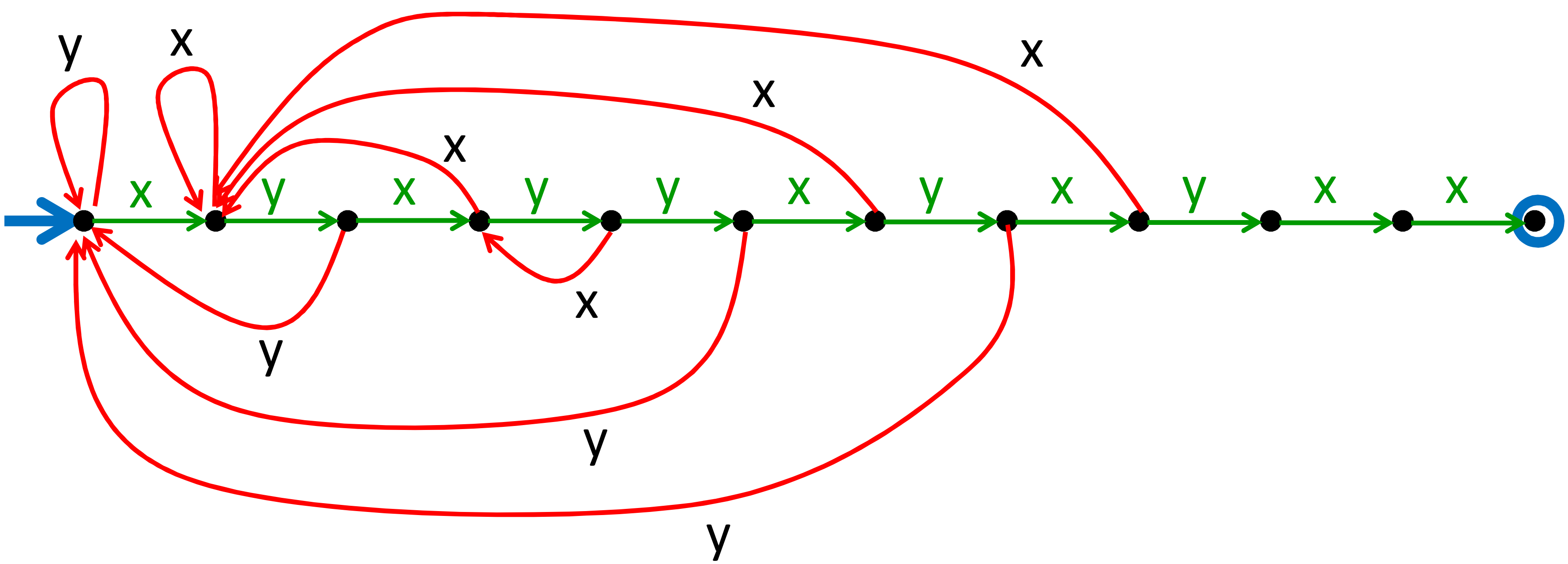
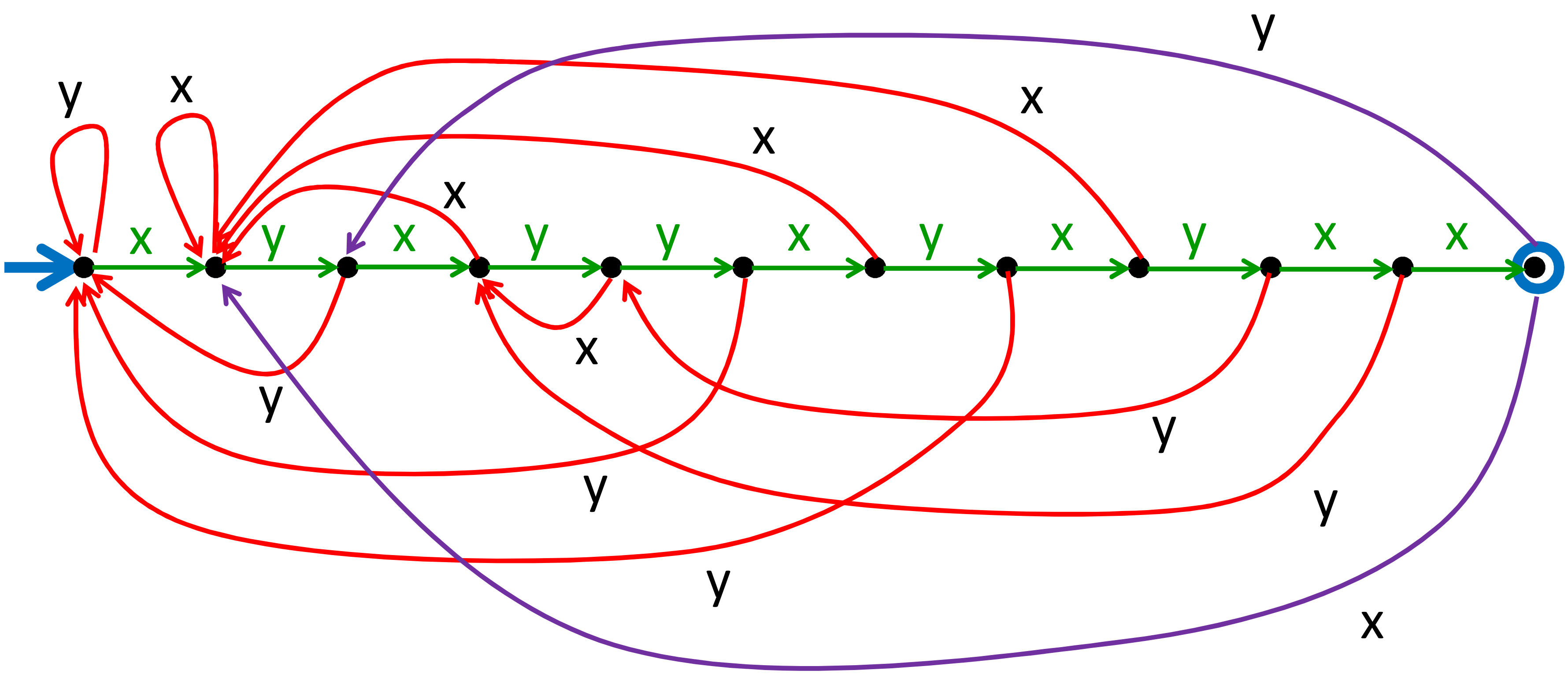


Building a DFA for the pattern

pattern **p**=x y x y y x y x y x x



Generalizing

- **Can search for arbitrary combinations of patterns**
  - **Not just a single pattern**
  - **Build NFA for pattern then convert to DFA ‘on the fly*’.**

    * Only add states when the input string actually needs to use them

    (Compare DFA constructed above with subset construction for the obvious NFA.)

DFAs ≡ NFAs ≡ Regular expressions

We have shown how to build an optimal DFA for every regular expression

- Build NFA
- Convert NFA to DFA using subset construction
- Minimize resulting DFA

Theorem: A language is recognized by a DFA (or NFA) if and only if it has a regular expression

You need to know this fact but we won't ask you anything about the "only if" direction from DFA/NFA to regular expression. For fun, we sketch the idea.

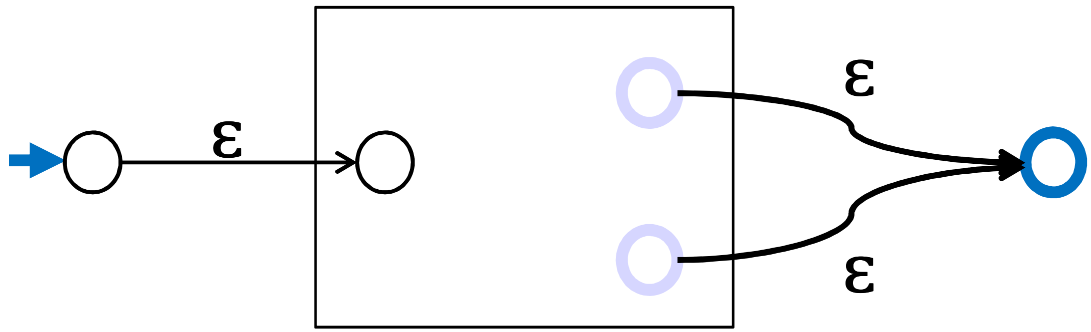
Generalized NFAs

- **Like NFAs but allow**
  - **Parallel edges**
  - **Regular Expressions as edge labels**

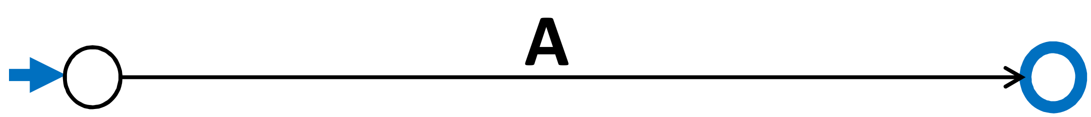
    NFAs already have edges labeled ε or ***a***
- **An edge labeled by A can be followed by reading a string of input chars that is in the language represented by A**
- **Defn: A string** x **is accepted iff there is a *path* from start to final state *labeled by a regular expression* whose language contains** x

Starting from an NFA

Add new start state and final state



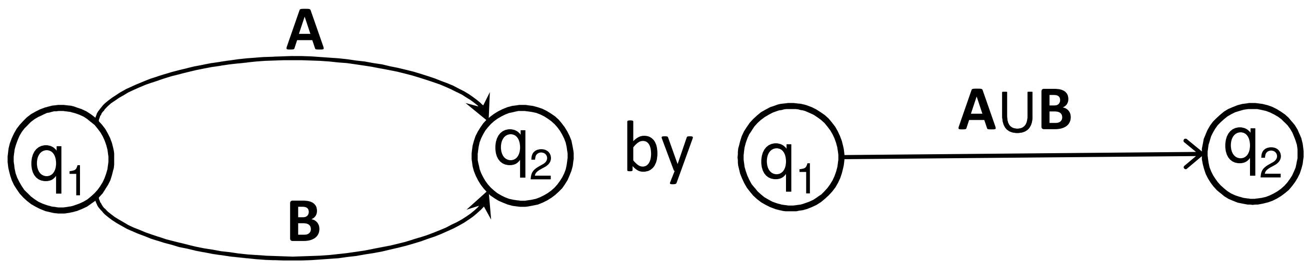
Then eliminate original states one by one, keeping the same language, until it looks like:



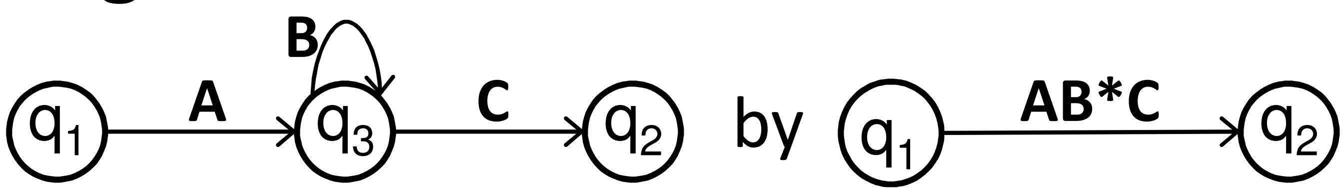
Final regular expression will be A

Only two simplification rules

- **Rule 1**: **For any two states q_1 and q_2 with parallel edges (possibly q_1=q_2), replace**



- **Rule 2**: **Eliminate non-start/final state q_3 by replacing all**

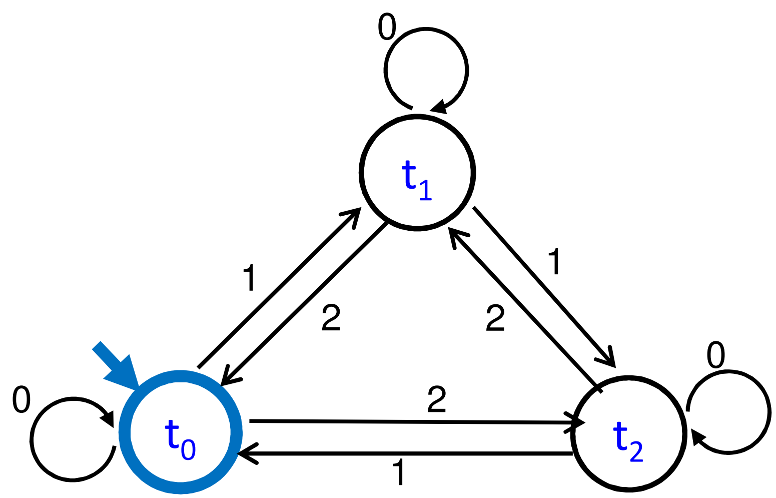


for *every* pair of states q_1, q_2 (even if q_1=q_2)

Converting an NFA to a regular expression

Consider the DFA for the mod 3 sum

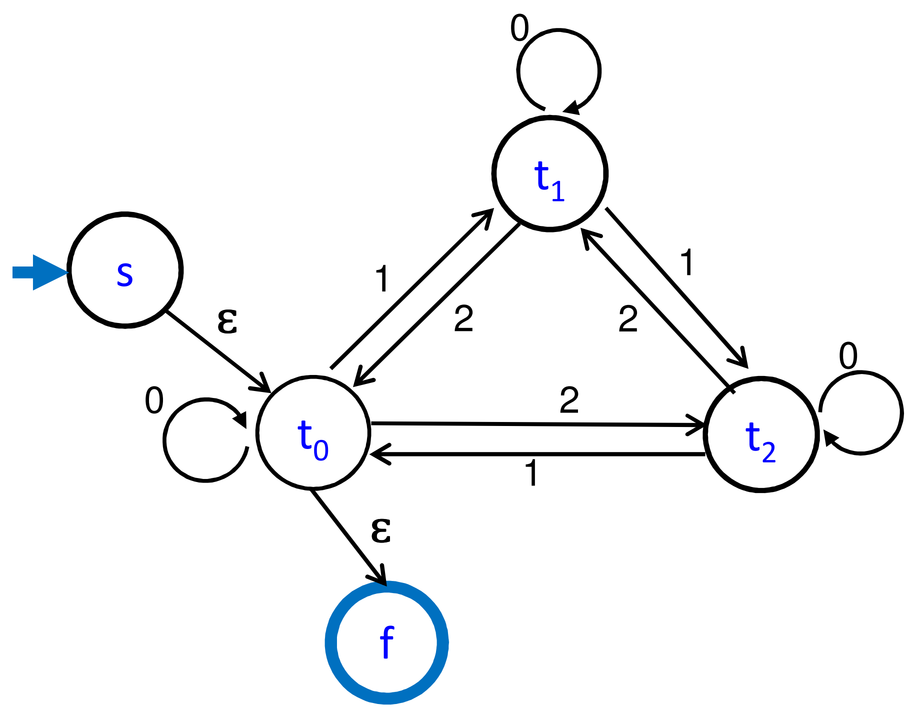
- Accept strings from {0,1,2}* where the digits mod 3 sum of the digits is 0



Splicing out a state t_1

Regular expressions to add to edges

$t_0 \to t_1 \to t_0$: 10*2
$t_0 \to t_1 \to t_2$: 10*1
$t_2 \to t_1 \to t_0$: 20*2
$t_2 \to t_1 \to t_2$: 20*1



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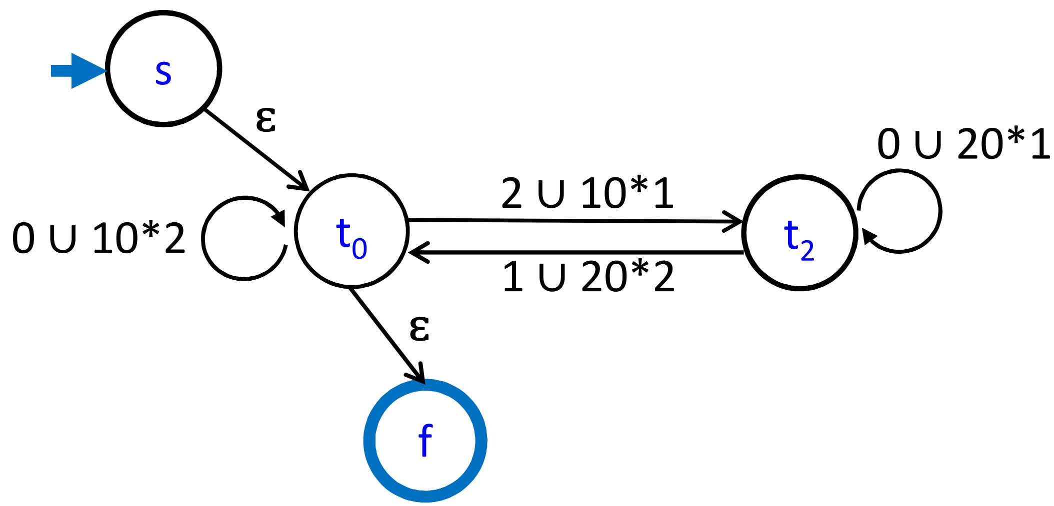
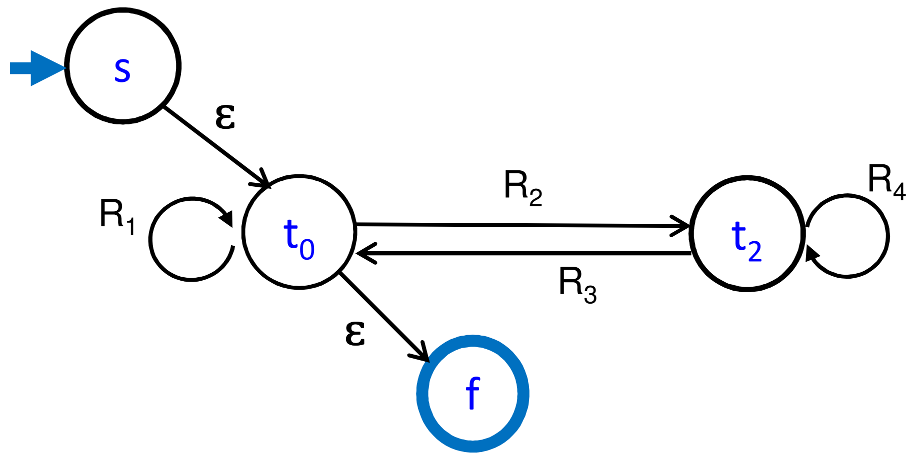
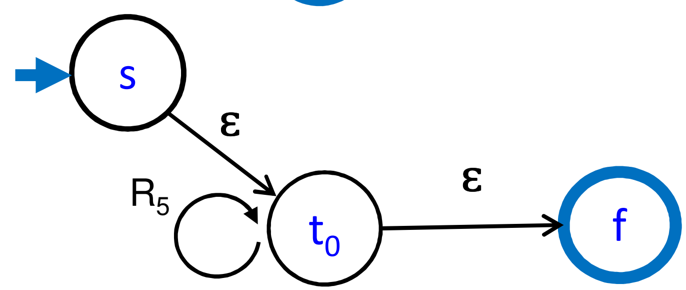


Splicing out state t_2 (and then t_0)

R_1: $0 \cup 10^*2$
R_2: $2 \cup 10^*1$
R_3: $1 \cup 20^*2$
R_4: $0 \cup 20^*1$



R_5: $R_1 \cup R_2R_4{}^*R_3$

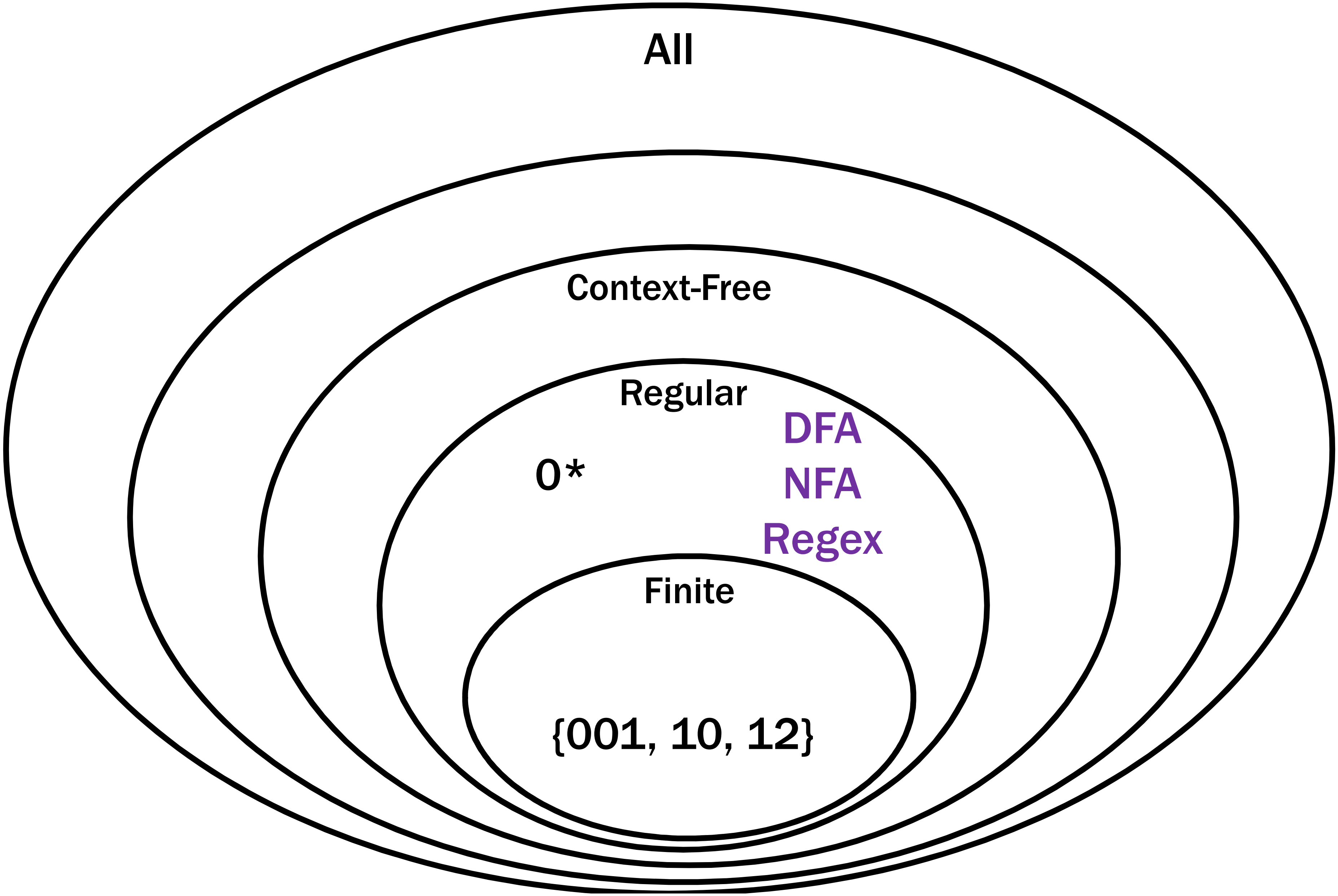


Final regular expression: $R_5{}^*$=
$(0 \cup 10^*2 \cup (2 \cup 10^*1)(0 \cup 20^*1)^*(1 \cup 20^*2))^*$

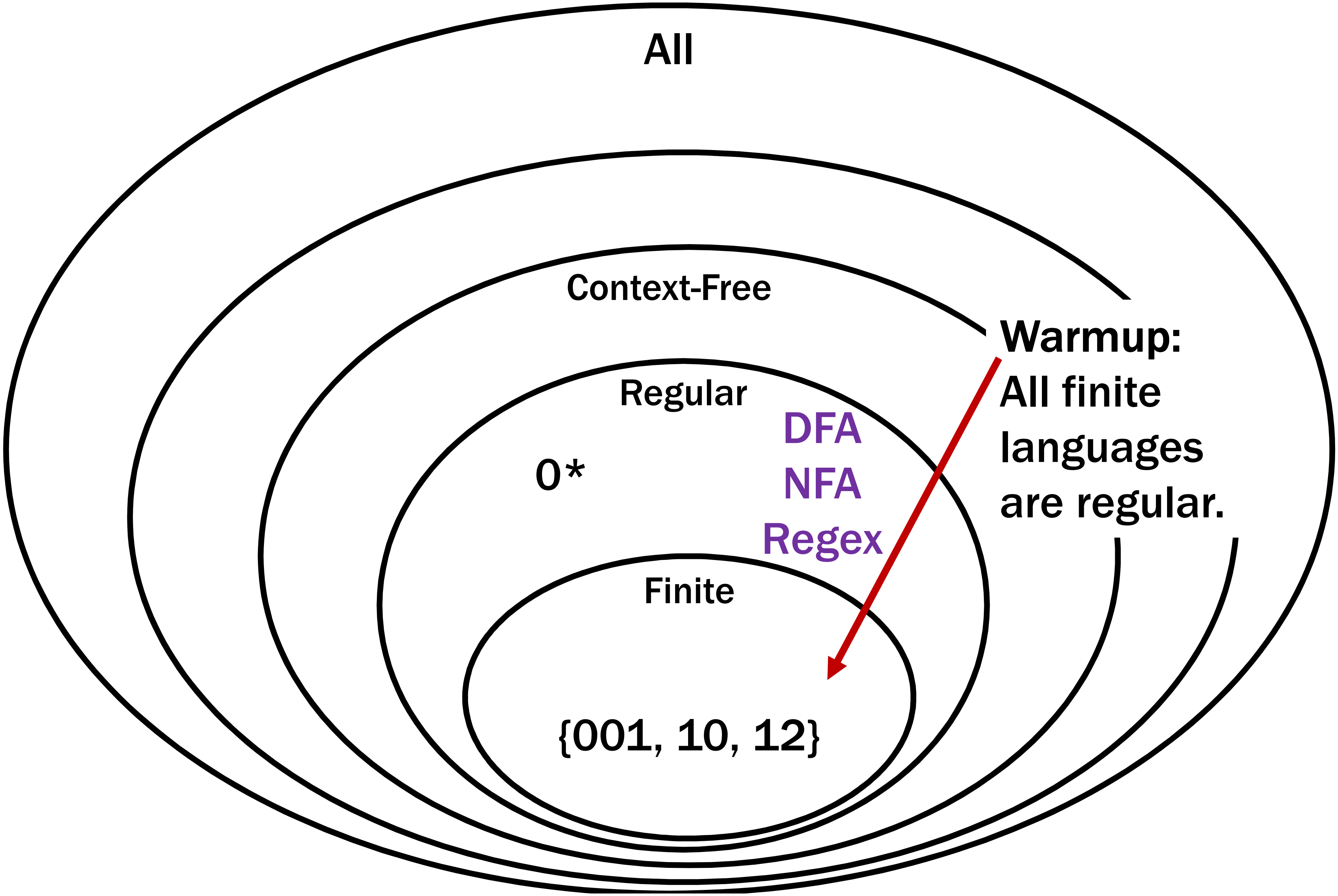
What languages have DFAs? CFGs?

All of them?

Languages and Representations!



Languages and Representations!



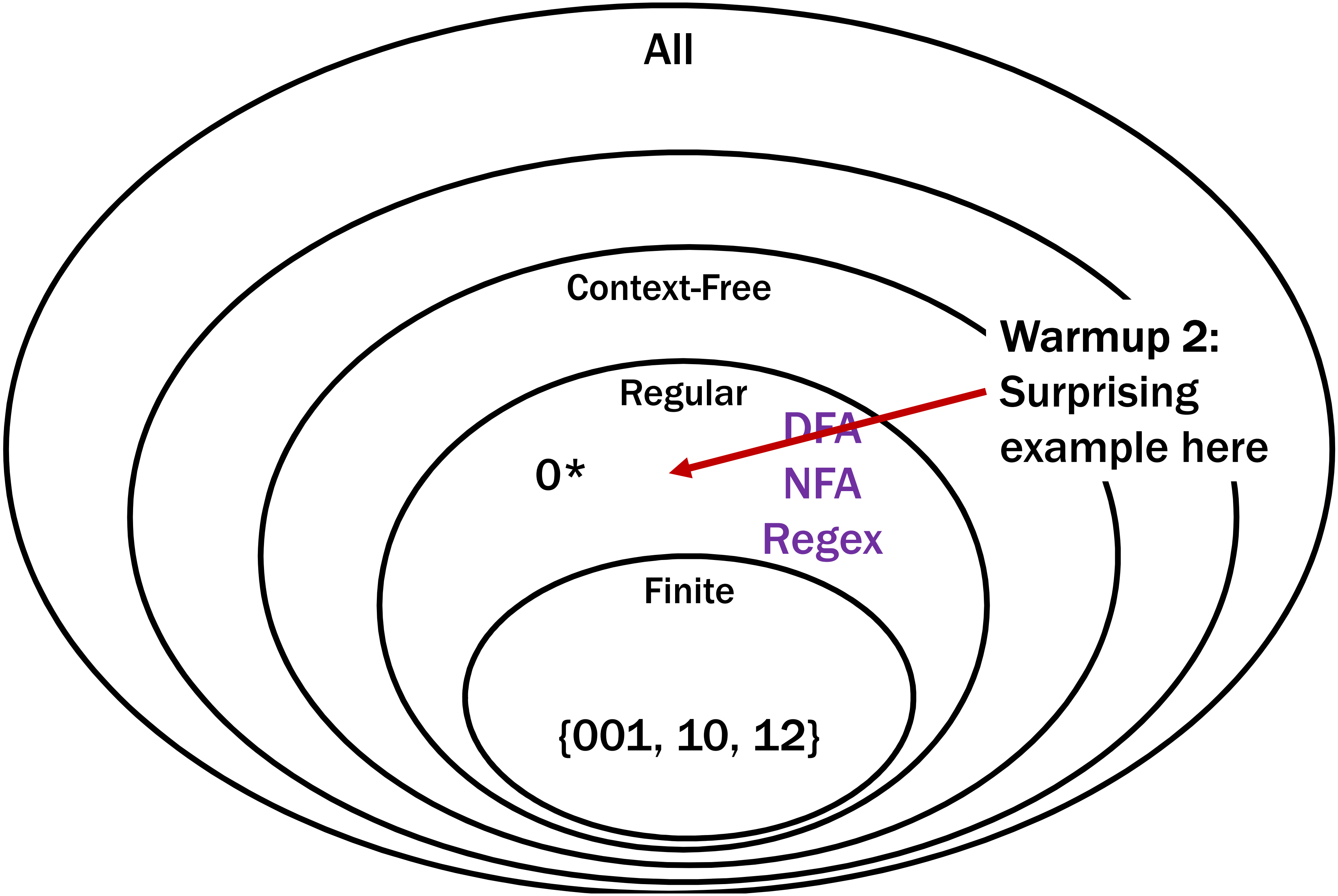
DFAs Recognize Any Finite Language

DFAs Recognize Any Finite Language

Construct a DFA for each string in the language.

Then, put them together using the union construction.

Languages and Machines!



An Interesting Infinite Regular Language

L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

L is infinite.

0, 00, 000, …

L is regular. How could this be?

(It seems to be comparing counts and counting seems hard for DFAs.)

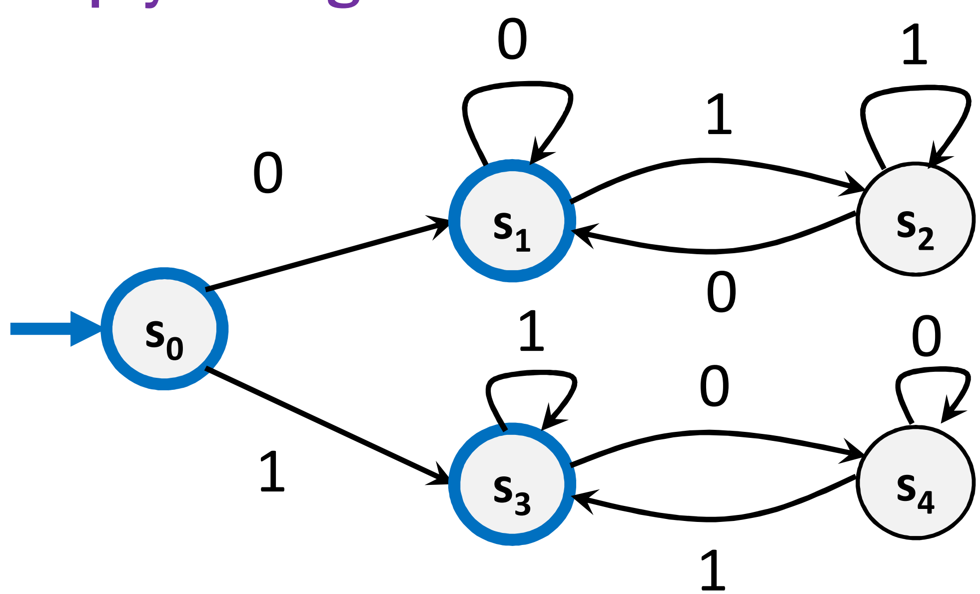
An Interesting Infinite Regular Language

L = {x∈ {0, 1}*: x has an equal number of substrings 01 and 10}.

L is infinite.

0, 00, 000, ...

L is regular. How could this be? It is just the set of binary strings that are empty or begin and end with the same character!



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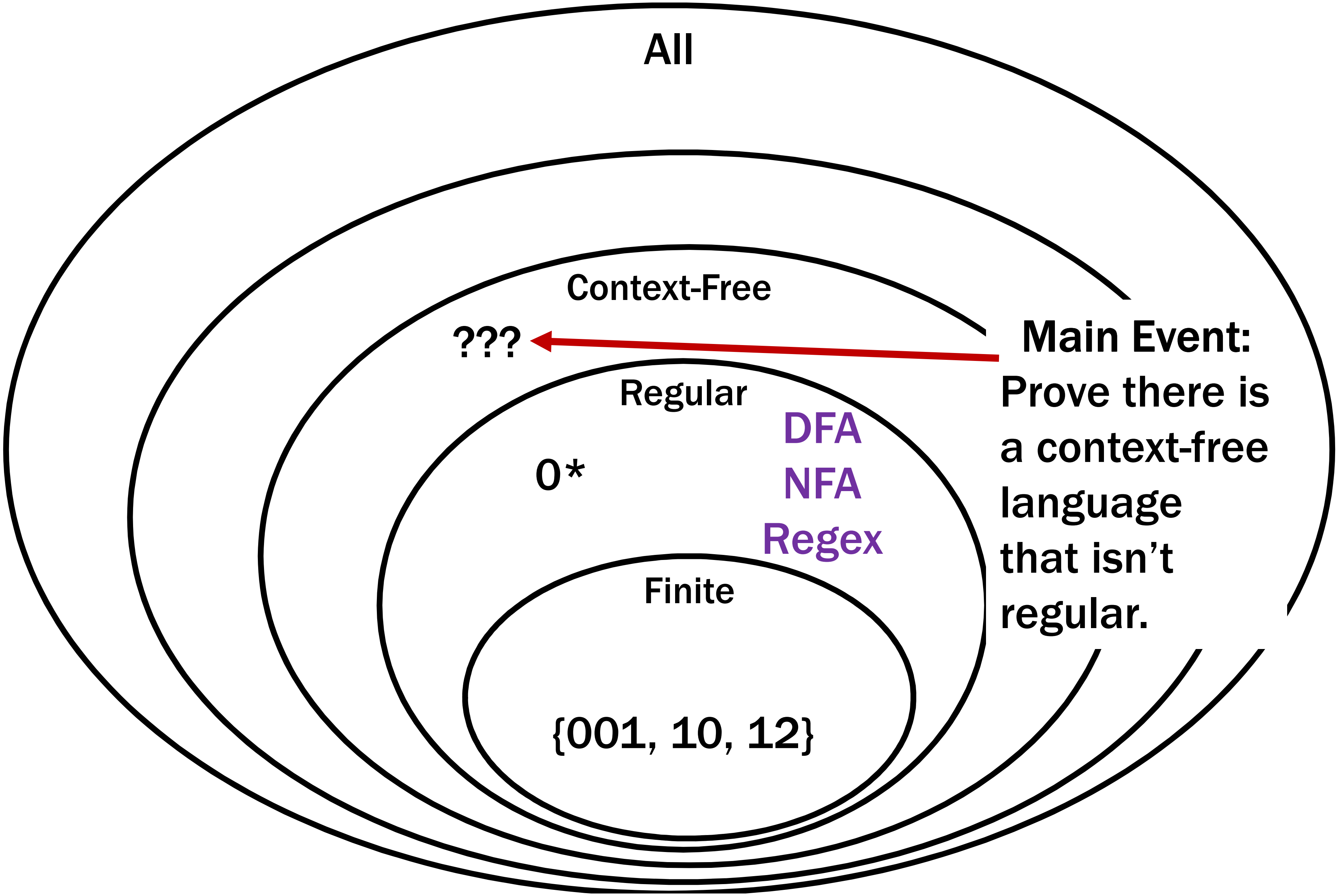

The language of “Binary Palindromes” is Context-Free

$$S \rightarrow \varepsilon \mid 0 \mid 1 \mid 0S0 \mid 1S1$$

Is the language of “Binary Palindromes” Regular ?

Is the language of “Binary Palindromes” Regular ?

Intuition (NOT A PROOF!):

- **Q**: What would a DFA need to keep track of to decide the language?
- **A**: It would need to keep track of the “first part” of the input in order to check the second part against it
  - …but there are an infinite # of possible first parts and we only have finitely many states.

B = {binary palindromes} can’t be recognized by any DFA

The general proof strategy is:

- Assume (for contradiction) that it’s possible.
- Therefore, some DFA (call it M) exists that recognizes B

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How can a DFA be “wrong”?

- when it accepts or rejects a string it shouldn’t.

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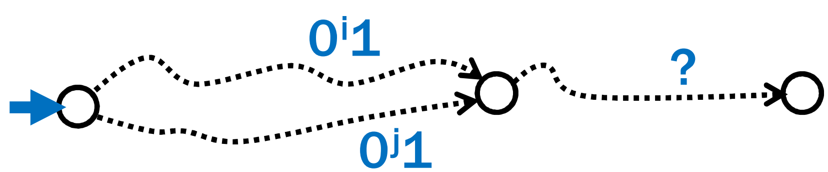
Key Idea 1: If two strings "collide" at any point, a DFA can no longer distinguish between them!



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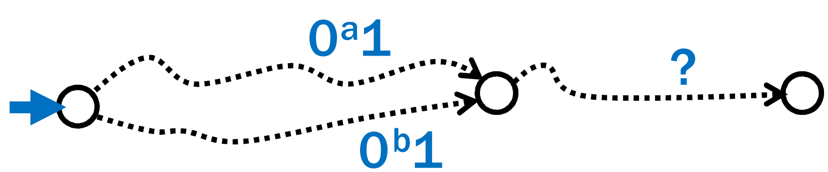

Key Idea 2: Our machine M has a finite number of states which means if we have infinitely many strings, two of them must collide!

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The general proof strategy is:

- Assume (for contradiction) that it's possible.
- Therefore, some DFA (call it M) exists that recognizes B
- We want to show M accepts or rejects a string it shouldn't.

We choose an INFINITE set S of "half strings" (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.

1______
01______
001______
0001______
00001______
....

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, recognizes B.

We show M accepts or rejects a string it shouldn't.

Consider S={1, 01, 001, 0001, 00001, ...} = $\{0^n1 : n \geq 0\}$.

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Suppose for contradiction that some DFA, M, recognizes B.

We show M accepts or rejects a string it shouldn't.

Consider $S=\{1, 01, 001, 0001, 00001, \ldots\} = \{0^n 1 : n \geq 0\}$.

Since there are finitely many states in M and infinitely many strings in S***, there exist strings*** $0^a 1 \in S$ ***and*** $0^b 1 \in S$ ***with*** $a \neq b$ ***that end in the same state of M.***

SUPER IMPORTANT POINT: You do not get to choose what a **and** b **are. Remember, we've proven they exist…we have to take the ones we're given!**

B = {binary palindromes} can't be recognized by any DFA

Suppose for contradiction that some DFA, M, accepts B.

We show M accepts or rejects a string it shouldn't.

Consider $S = \{0^n1 : n \geq 0\}$.

Since there are finitely many states in M and infinitely many strings in S**, there exist strings** $0^a1 \in S$ **and** $0^b1 \in S$ **with** $a \neq b$ **that end in the same state of M.**

Now, consider appending 0^a ***to both strings.***

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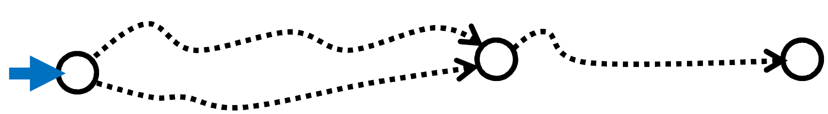

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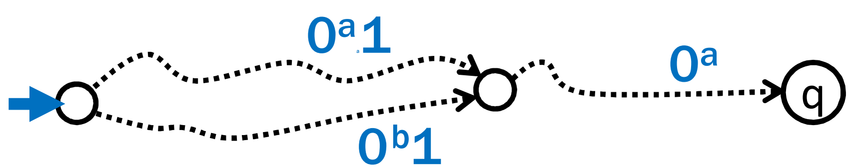

Then, since 0^a1 ***and*** 0^b1 ***end in the same state,*** 0^a10^a ***and*** 0^b10^a ***also end in the same state, call it*** q***. But then M must make a mistake:*** q ***needs to be an accept state since*** $0^a10^a \in$ ***B, but then M would accept*** $0^b10^a \notin$ ***B which is an error.***

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Consider $S = \{0^n1 : n \geq 0\}$.

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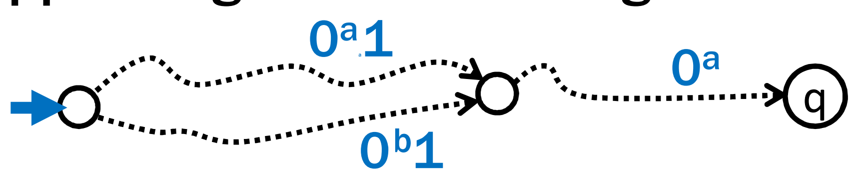

Then, since 0^a1 **and** 0^b1 **end in the same state,** 0^a10^a **and** 0^b10^a **also end in the same state, call it** q**. But then M must make a mistake:** q **needs to be an accept state since** $0^a10^a \in$ **B, but then M would accept** $0^b10^a \notin$ **B which is an error.**

This is a contradiction, since we assumed that M recognizes B. Since M was arbitrary, there is no DFA that recognizes B.

Showing that a Language L is not regular

1. "Suppose for contradiction that some DFA M recognizes L."
2. Consider an INFINITE set S of "half strings" (which we intend to complete later). It is imperative that for *every pair* of strings in our set there is an "accept" completion that the two strings DO NOT SHARE.
3. "Since S is infinite and M has finitely many states, there must be two strings s_a and s_b in S for some $s_a \neq s_b$ that end up at the same state of M."
4. Consider appending the (correct) completion to each of the two strings.
5. "Since s_a and s_b both end up at the same state of M, and we appended the same string t, both $s_a t$ and $s_b t$ end at the same state q of M. Since $s_a t \in L$ and $s_b t \notin L$, M does not recognize L."
6. "Since M was arbitrary, no DFA recognizes L."