CSE 311: Foundations of Computing

Lecture 18: Structural Induction, Regular expressions

Midterm

range 37-99

Median grade 81

90's 65

80's 75

70's 69

60's 39

<60 ~20

Recursive Definitions of Sets: General Form

Recursive definition

– ***Basis step:*** Some specific elements are in S
– ***Recursive step:*** Given some existing named elements in S some new objects constructed from these named elements are also in S.
– ***Exclusion rule***: Every element in S follows from the basis step and a finite number of recursive steps

Structural Induction

Basis → $0 \in \mathbb{N}$

Recursive If $k \in \mathbb{N}$ then $k+1 \in \mathbb{N}$

How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the ***Basis step***

Inductive Hypothesis: Assume that P is true for some arbitrary values of ***each*** of the existing named elements mentioned in the ***Recursive step***

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the ***Recursive step*** using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma$ (ε is the empty string w/ no chars)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Functions on Recursively Defined Sets (on Σ^*)

Length:

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$

Number of c's in a string:

$\#_c(\varepsilon) = 0$

$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

Claim: $\mathrm{len}(x \bullet y) = \mathrm{len}(x) + \mathrm{len}(y)$ **for all** $x, y \in \Sigma^*$

Let $P(y)$ **be** "$\mathrm{len}(x \bullet y) = \mathrm{len}(x) + \mathrm{len}(y)$ for all $x \in \Sigma^*$".
We prove $P(y)$ **for all** $y \in \Sigma^*$ **by structural induction.**

Claim: len($x \bullet y$) = len(x) + len(y) **for all** $x,y \in \Sigma^*$

Let $P(y)$ **be** "len($x \bullet y$) = len(x) + len(y) for all $x \in \Sigma^*$".
We prove $P(y)$ **for all** $y \in \Sigma^*$ **by structural induction.**

Base Case: $y = \varepsilon$. **For any** $x \in \Sigma^*$, len($x \bullet \varepsilon$) = len(x) = len(x) + len(ε)
since len(ε)=0. **Therefore** $P(\varepsilon)$ **is true**

Claim: $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ for all $x,y \in \Sigma^*$

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Base Case: $y = \varepsilon$. **For any** $x \in \Sigma^*$, $\text{len}(x \bullet \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ **since** $\text{len}(\varepsilon)=0$. **Therefore** $P(\varepsilon)$ **is true**

Inductive Hypothesis: **Assume that** $P(w)$ **is true for some arbitrary** $w \in \Sigma^*$

Inductive Step: **Goal: Show that** $P(wa)$ **is true for every** $a \in \Sigma$

Claim: $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ for all $x, y \in \Sigma^*$

Let $P(y)$ **be** "$\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$".
We prove $P(y)$ **for all** $y \in \Sigma^*$ **by structural induction.**

Base Case: $y = \varepsilon$. **For any** $x \in \Sigma^*$, $\text{len}(x \bullet \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$ **since** $\text{len}(\varepsilon) = 0$. **Therefore** $P(\varepsilon)$ **is true**

Inductive Hypothesis: Assume that $P(w)$ **is true for some arbitrary** $w \in \Sigma^*$

Inductive Step: $\boxed{\textbf{Goal: Show that } P(wa) \textbf{ is true for every } a \in \Sigma}$

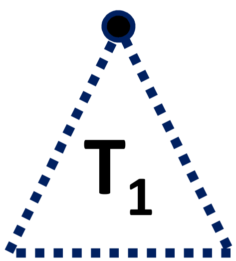
Let $a \in \Sigma$. **Let** $x \in \Sigma^*$. **Then** $\text{len}(x \bullet wa) = \text{len}((x \bullet w)a)$ **by defn of** $\bullet$
$= \text{len}(x \bullet w) + 1$ **by defn of len**
$= \text{len}(x) + \text{len}(w) + 1$ **by I.H.**
$= \text{len}(x) + \text{len}(wa)$ **by defn of len**

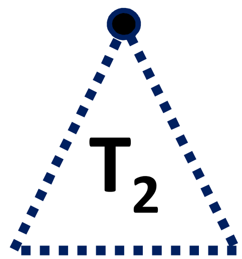
Therefore $\text{len}(x \bullet wa) = \text{len}(x) + \text{len}(wa)$ **for all** $x \in \Sigma^*$, **so** $P(wa)$ **is true.**

So, by induction $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ **for all** $x, y \in \Sigma^*$

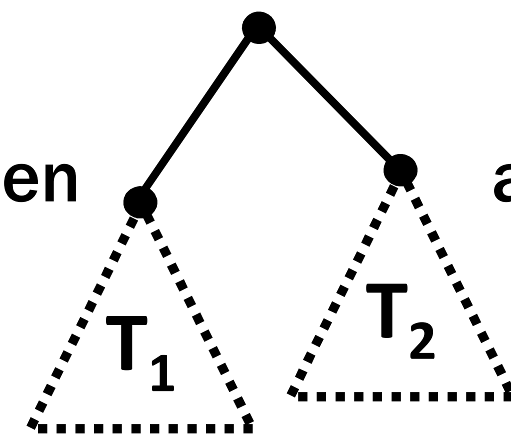
Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**

If 

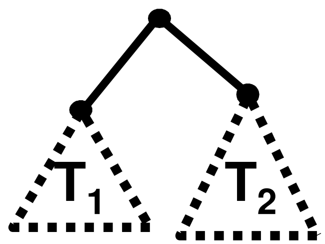
and 

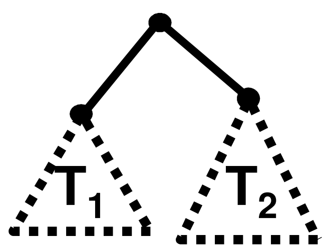
are rooted binary trees,

then 

also is a rooted binary tree.

Defining Functions on Rooted Binary Trees

- size(•) = 1
- size (

) = 1 + size(**T_1**) + size(**T_2**)
- height(•) = 0
- height (

)=1 + max{height(**T_1**), height(**T_2**)}

Claim: **For every rooted binary tree T,** $\text{size}(\mathbf{T}) \leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

Claim: For every rooted binary tree T, size(**T**) $\leq 2^{\text{height}(\mathbf{T})+1} - 1$

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1. **Let** P(**T**) **be** "size(T) $\leq 2^{\text{height}(T)+1} - 1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
2. **Base Case:** size(•)=1, height(•)=0 **and** $1=2^1-1=2^{0+1}-1$ **so** P(•) **is true.**

Claim: For every rooted binary tree **T**, $\text{size}(\mathbf{T}) \leq 2^{\text{height}(\mathbf{T})+1} - 1$

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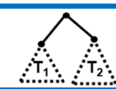
1. **Let** $P(\mathbf{T})$ **be** "$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$". **We prove** $P(\mathbf{T})$ **for all rooted binary trees T by structural induction.**
2. **Base Case:** $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ **and** $1=2^1-1=2^{0+1}-1$ **so** $P(\bullet)$ **is true.**
3. **Inductive Hypothesis: Suppose that** $P(\mathbf{T_1})$ **and** $P(\mathbf{T_2})$ **are true for some rooted binary trees** $\mathbf{T_1}$ **and** $\mathbf{T_2}$.
4. **Inductive Step:**

Goal: Prove P().

Claim: For every rooted binary tree T, size(**T**) $\leq 2^{\text{height}(\mathbf{T})+1} - 1$

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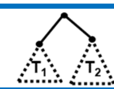
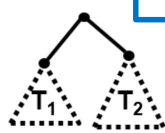
1. **Let** P(**T**) **be** "size(T) $\leq 2^{\text{height}(T)+1}-1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
2. **Base Case:** size(•)=1, height(•)=0 **and** $1=2^1-1=2^{0+1}-1$ **so** P(•) **is true.**
3. **Inductive Hypothesis: Suppose that** P(**T$_1$**) **and** P(**T$_2$**) **are true for some rooted binary trees T$_1$ and T$_2$.**
4. **Inductive Step:** **Goal: Prove** P().

By defn, size() $=1+\text{size}(\mathbf{T_1})+\text{size}(\mathbf{T_2})$ by defn of size

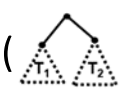
$\leq 1+2^{\text{height}(\mathbf{T_1})+1}-1+2^{\text{height}(\mathbf{T_2})+1}-1$

by IH for $\mathbf{T_1}$ and $\mathbf{T_2}$

$= 2^{\text{height}(\mathbf{T_1})+1}+2^{\text{height}(\mathbf{T_2})+1}-1$

$= 2(\max(2^{\text{height}(T_1)+1}, 2^{\text{height}(T_2)+1}))$

$\leq 2(2^{\max(\text{height}(\mathbf{T_1}),\text{height}(\mathbf{T_2}))+1})-1$

$= 2(2^{\text{height}(}$  $^{)}) - 1 = 2^{\text{height}(}$  $^{)+1} - 1$

which is what we wanted to show.

5. **So, the** P(T) **is true for all rooted bin. trees by structural induction.**

Languages: Sets of Strings

- Sets of strings that satisfy special properties are called *languages*. Examples:
  - English sentences
  - Syntactically correct Java/C/C++ programs
  - $\Sigma^* =$ All strings over alphabet Σ $\{0,1\}^*$ binary strings
  - Palindromes over Σ
  - Binary strings that don't have a 0 after a 1
  - Legal variable names. keywords in Java/C/C++
  - Binary strings with an equal # of 0's and 1's

Regular Expressions

Regular expressions over Σ

- **Basis:**
  - $\varnothing$, ε are regular expressions
  - ***a*** is a regular expression for any $a \in \Sigma$
- **Recursive step:**
  - If **A** and **B** are regular expressions then so are:
    - **(A $\cup$ B)**
    - **(AB)**
    - **A***

Each Regular Expression is a "pattern"

ϕ doesn't match anything

ε matches the **empty string** $\{\varepsilon\}$

a matches the one character string *a* $\{a\}$

(A ∪ B) matches all strings that either **A** matches or **B** matches (or both) $((a \cup (a a)) b)$ $\{ab, aab\}$

(AB) matches all strings that have a first part that **A** matches followed by a second part that **B** matches

A* matches all strings that have any number of strings (even 0) that **A** matches, one after another

Examples

001*

{00, 001, 0011, 00111, ...

0*1*

{ε, 0, 1, 00, 011, ...

Examples

001*

{00, 001, 0011, 00111, ...}

0*1*

Any number of 0's followed by any number of 1's

Examples

$(0 \cup 1)\, 0\, (0 \cup 1)\, 0$

$\{0000,\ 0010,\ 1000,\ 1010\}$

$(0^*1^*)^*$

$(0 \cup 1)^*$

Examples

$(0 \cup 1)\ 0\ (0 \cup 1)\ 0$

{0000, 0010, 1000, 1010}

$(0^*1^*)^*$

All binary strings

Examples

$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$

Any binary string that contains 0110

$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$

Examples

$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$

Binary strings that contain “0110”

$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$

Binary strings that begin with pairs of characters followed by “01010” or “10001”

Regular Expressions in Practice

- Used to define the "tokens": e.g., legal variable names, keywords in programming languages and compilers
- Used in **`grep`**, a program that does pattern matching searches in UNIX/LINUX
- Pattern matching using regular expressions is an essential feature of PHP Javascript.
- We can use regular expressions in programs to process strings!

Regular Expressions in Java

- Pattern p = Pattern.compile("a*b");
- Matcher m = p.matcher("aaaaab");
- boolean b = m.matches();

  `[01]` a 0 or a 1 `^` start of string `$` end of string

  `[0-9]` any single digit `\.` period `\,` comma `\-` minus

  `.` any single character

ab	a followed by b	**(AB)**	Concatenation
(a|b)	a or b	$(\mathbf{A} \cup \mathbf{B})$	union
a**?**	zero or one of a	$(\mathbf{A} \cup \varepsilon)$	
a*****	zero or more of a	$\mathbf{A}^*$	Kleene Star
a**+**	one or more of a	$\mathbf{AA}^*$	

- e.g. `^[\-+]?[0-9]*(\.|\,)?[0-9]+$`

  General form of decimal number e.g. 9.12 or -9,8 (Europe)