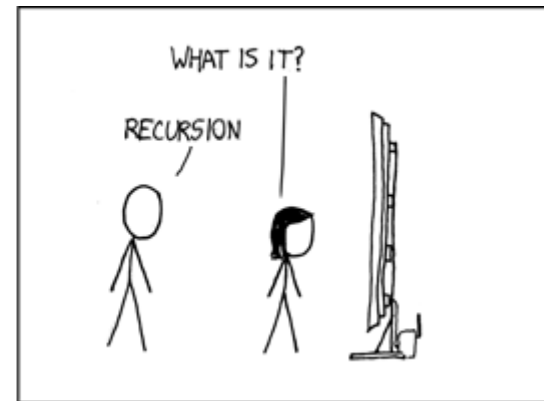
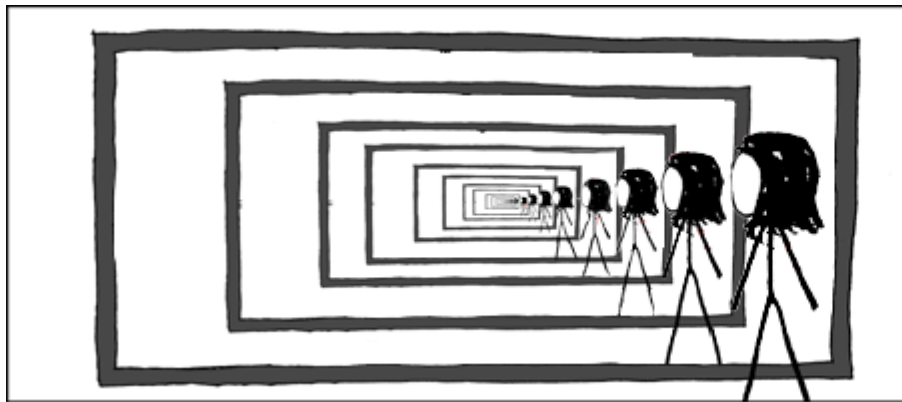


CSE 311: Foundations of Computing

Lecture 17: Recursively Defined Sets & Structural Induction



Recursive Definition of Sets

Recursive definition of set S

- **Basis Step:** $0 \in S$
- **Recursive Step:** If $x \in S$, then $x + 2 \in S$
- **Exclusion Rule:** Every element in S follows from the basis step and a finite number of recursive steps.

We need the exclusion rule because otherwise $S = \mathbb{N}$ would satisfy the other two parts. However, we won't always write it down on these slides.

Recursive Definitions of Sets

Basis: $6 \in S, 15 \in S$

Recursive: If $x, y \in S$, then $x+y \in S$

Basis: $[1, 1, 0] \in S, [0, 1, 1] \in S$

Recursive: If $[x, y, z] \in S$, then $[\alpha x, \alpha y, \alpha z] \in S$ for any $\alpha \in \mathbb{R}$

If $[x_1, y_1, z_1] \in S$ and $[x_2, y_2, z_2] \in S$, then
 $[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

Number of form 3^n for $n \geq 0$:

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Number of form 3^n for $n \geq 0$:

Basis: $1 \in S$

Recursive: If $x \in S$, then $3x \in S$.

Recursive Definitions of Sets: General Form

Recursive definition

- ***Basis step:*** Some specific elements are in S
- ***Recursive step:*** Given some existing named elements in S some new objects constructed from these named elements are also in S .
- ***Exclusion rule:*** Every element in S follows from the basis step and a finite number of recursive steps

Strings

- An *alphabet* Σ is any finite set of characters
- The set Σ^* of *strings* over the alphabet Σ is defined by
 - **Basis:** $\varepsilon \in \Sigma$ (ε is the empty string w/ no chars)
 - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same backwards and forwards

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$

All Binary Strings with no 1's before 0's

All Binary Strings with no 1's before 0's

Basis:

$\varepsilon \in S$

Recursive:

If $x \in S$, then $0x \in S$

If $x \in S$, then $x1 \in S$

Functions on Recursively Defined Sets (on Σ^*)

Length:

$$\text{len}(\varepsilon) = 0$$

$$\text{len}(wa) = 1 + \text{len}(w) \text{ for } w \in \Sigma^*, a \in \Sigma$$

Reversal:

$$\varepsilon^R = \varepsilon$$

$$(wa)^R = aw^R \text{ for } w \in \Sigma^*, a \in \Sigma$$

Concatenation:

$$x \bullet \varepsilon = x \text{ for } x \in \Sigma^*$$

$$x \bullet wa = (x \bullet w)a \text{ for } x \in \Sigma^*, a \in \Sigma$$

Number of c 's in a string:

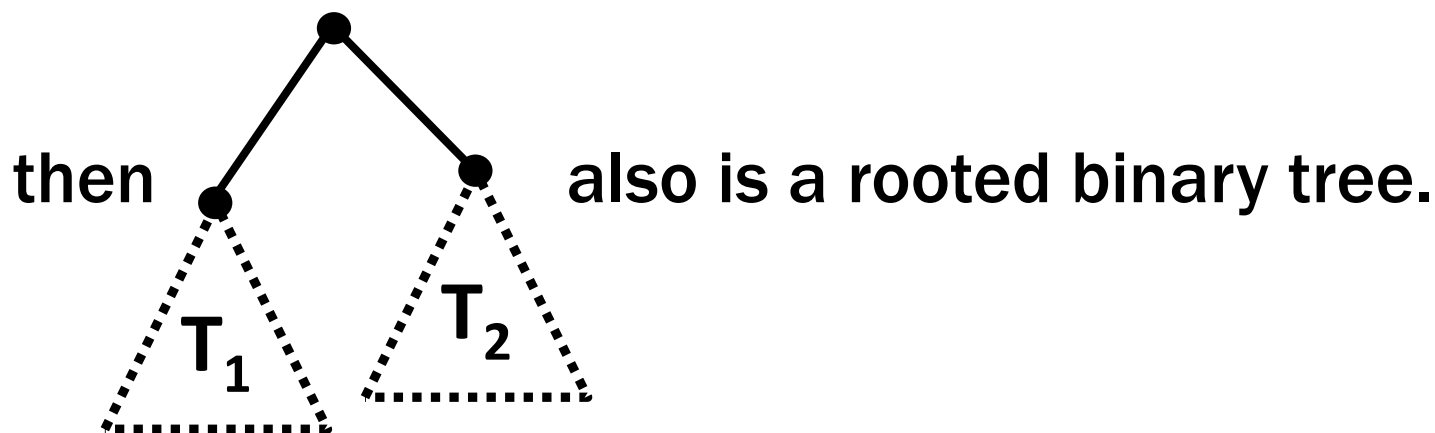
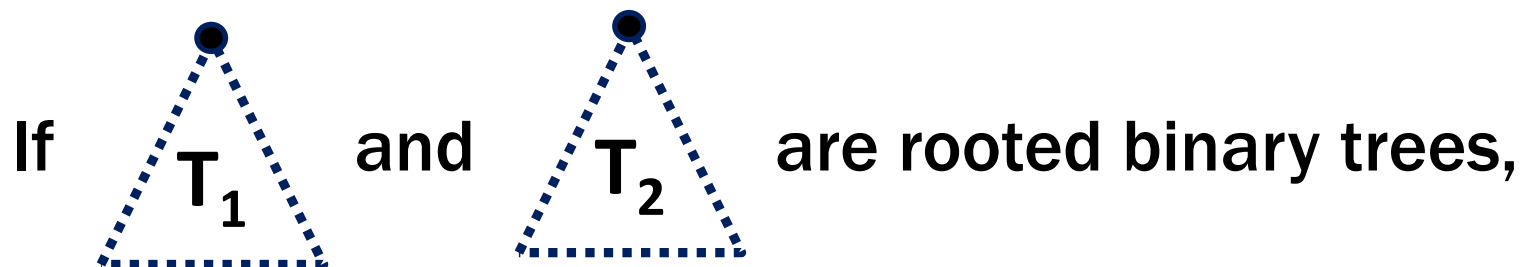
$$\#_c(\varepsilon) = 0$$

$$\#_c(wc) = \#_c(w) + 1 \text{ for } w \in \Sigma^*$$

$$\#_c(wa) = \#_c(w) \text{ for } w \in \Sigma^*, a \in \Sigma, a \neq c$$

Rooted Binary Trees

- **Basis:** • is a rooted binary tree
- **Recursive step:**



Structural Induction

How to prove $\forall x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step*

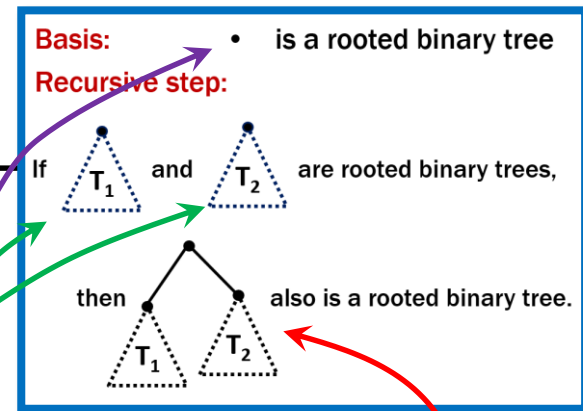
Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step*

Inductive Step: Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall x \in S, P(x)$

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Conclude that $\forall x \in S, P(x)$

Structural Induction vs. Ordinary Induction

Ordinary induction is a special case of structural induction:

Recursive definition of \mathbb{N}

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Define $Q(n)$ to be “for all $x \in S$ that can be constructed in at most n recursive steps, $P(x)$ is true.”

Using Structural Induction

- Let S be given by...
 - **Basis:** $6 \in S$; $15 \in S$;
 - **Recursive:** if $x, y \in S$ then $x + y \in S$.

Claim: Every element of S is divisible by 3.

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1. Let $P(x)$ be “ $3 \mid x$ ”. We prove that $P(x)$ is true for all $x \in S$ by structural induction.
2. Base Case: $3 \mid 6$ and $3 \mid 15$ so $P(6)$ and $P(15)$ are true

Basis: $6 \in S$; $15 \in S$;

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3. Inductive Hypothesis: Suppose that $P(x)$ and $P(y)$ are true for some arbitrary $x, y \in S$
4. Inductive Step: Goal: Show $P(x+y)$

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Since $P(x)$ is true, $3 \mid x$ and so $x=3m$ for some integer m and since $P(y)$ is true, $3 \mid y$ and so $y=3n$ for some integer n .

Therefore $x+y=3m+3n=3(m+n)$ and thus $3 \mid (x+y)$.

Hence $P(x+y)$ is true.

5. Therefore by induction $3 \mid x$ for all $x \in S$.

Basis: $6 \in S$; $15 \in S$;

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Claim: $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ for all $x, y \in \Sigma^*$

Let $P(y)$ be “ $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ for all $x \in \Sigma^*$ ”.

We prove $P(y)$ for all $y \in \Sigma^*$ by structural induction.

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Base Case: $y = \varepsilon$. For any $x \in \Sigma^*$, $\text{len}(x \bullet \varepsilon) = \text{len}(x) = \text{len}(x) + \text{len}(\varepsilon)$
since $\text{len}(\varepsilon) = 0$. Therefore $P(\varepsilon)$ is true

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 $w \in \Sigma^*$

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Let $a \in \Sigma$. Let $x \in \Sigma^*$. Then $\text{len}(x \bullet wa) = \text{len}((x \bullet w)a)$ by defn of \bullet
 $= \text{len}(x \bullet w) + 1$ by defn of len
 $= \text{len}(x) + \text{len}(w) + 1$ by I.H.
 $= \text{len}(x) + \text{len}(wa)$ by defn of len

Therefore $\text{len}(x \bullet wa) = \text{len}(x) + \text{len}(wa)$ for all $x \in \Sigma^*$, so $P(wa)$ is true.

So, by induction $\text{len}(x \bullet y) = \text{len}(x) + \text{len}(y)$ for all $x, y \in \Sigma^*$

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1. Let $P(T)$ be “ $\text{size}(T) \leq 2^{\text{height}(T)+1}-1$ ”. We prove $P(T)$ for all rooted binary trees T by structural induction.
2. Base Case: $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ and $1=2^1-1=2^{0+1}-1$ so $P(\bullet)$ is true.

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4. Inductive Step:

Goal: Prove $P(\text{ } \begin{array}{c} \triangle \\ / \quad \backslash \\ \triangle_{T_1} \quad \triangle_{T_2} \end{array} \text{ })$.

By defn, $\text{size}(\text{ } \begin{array}{c} \triangle \\ / \quad \backslash \\ \triangle_{T_1} \quad \triangle_{T_2} \end{array} \text{ }) = 1 + \text{size}(T_1) + \text{size}(T_2)$

$$\leq 1 + 2^{\text{height}(T_1)+1} - 1 + 2^{\text{height}(T_2)+1} - 1$$

by IH for T_1 and T_2

$$\leq 2^{\text{height}(T_1)+1} + 2^{\text{height}(T_2)+1} - 1$$

$$\leq 2(2^{\max(\text{height}(T_1), \text{height}(T_2))+1}) - 1$$

$$\leq 2(2^{\text{height}(\text{ } \begin{array}{c} \triangle \\ / \quad \backslash \\ \triangle_{T_1} \quad \triangle_{T_2} \end{array} \text{ })}) - 1 \leq 2^{\text{height}(\text{ } \begin{array}{c} \triangle \\ / \quad \backslash \\ \triangle_{T_1} \quad \triangle_{T_2} \end{array} \text{ })+1} - 1$$

which is what we wanted to show.

5. So, the $P(T)$ is true for all rooted bin. trees by structural induction.