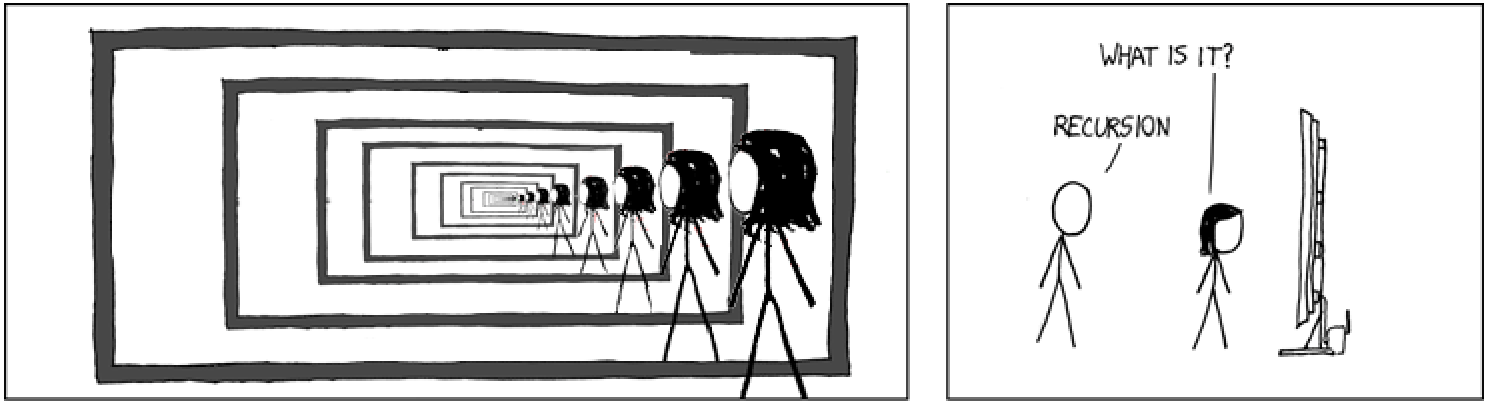
CSE 311: Foundations of Computing

Lecture 17: Recursively Defined Sets & Structural Induction



Recursive Definition of Sets

Recursive definition of set S

- **Basis Step:** $0 \in S$
- **Recursive Step: If** $x \in S$, **then** $x + 2 \in S$
- **Exclusion Rule: Every element in** S **follows from the basis step and a finite number of recursive steps.**

We need the exclusion rule because otherwise $S = \mathbb{N}$ would satisfy the other two parts. However, we won't always write it down on these slides.

Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$
Recursive: **If** $x,y \in S$, **then** $x+y \in S$

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$
Recursive: **If** $[x, y, z] \in S$, **then** $[\alpha x, \alpha y, \alpha z] \in S$ **for any** $\alpha \in \mathbb{R}$
If $[x_1, y_1, z_1] \in S$ **and** $[x_2, y_2, z_2] \in S$, **then**
$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

Number of form 3^n **for** $n \geq 0$:

Recursive Definitions of Sets

Basis: $6 \in S$, $15 \in S$
Recursive: **If** $x,y \in S$, **then** $x+y \in S$

Basis: $[1, 1, 0] \in S$, $[0, 1, 1] \in S$
Recursive: **If** $[x, y, z] \in S$, **then** $[\alpha x, \alpha y, \alpha z] \in S$ **for any** $\alpha \in \mathbb{R}$
If $[x_1, y_1, z_1] \in S$ **and** $[x_2, y_2, z_2] \in S$, **then**
$[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$.

Number of form 3^n **for** $n \geq 0$:
Basis: $1 \in S$
Recursive: If $x \in S$, **then** $3x \in S$.

Recursive Definitions of Sets: General Form

Recursive definition

- ***Basis step:*** Some specific elements are in S
- ***Recursive step:*** Given some existing named elements in S some new objects constructed from these named elements are also in S.
- ***Exclusion rule***: Every element in S follows from the basis step and a finite number of recursive steps

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ* of *strings* over the alphabet Σ is defined by
  - Basis: $\varepsilon \in \Sigma$ (ε is the empty string w/ no chars)
  - Recursive: if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes

Palindromes are strings that are the same backwards and forwards

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$

All Binary Strings with no 1's before 0's

All Binary Strings with no 1's before 0's

Basis:

$\varepsilon \in S$

Recursive:

If $x \in S$, **then** $0x \in S$

If $x \in S$, **then** $x1 \in S$

Functions on Recursively Defined Sets (on Σ^*)

Length:

$\text{len}(\varepsilon) = 0$

$\text{len}(wa) = 1 + \text{len}(w)$ for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$

$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x \in \Sigma^*$, $a \in \Sigma$

Number of c's in a string:

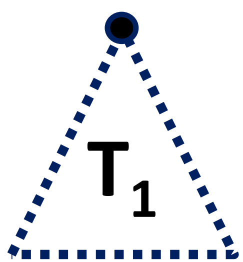
$\#_c(\varepsilon) = 0$

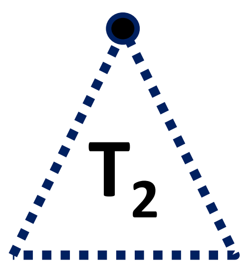
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$

$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*$, $a \in \Sigma$, $a \neq c$

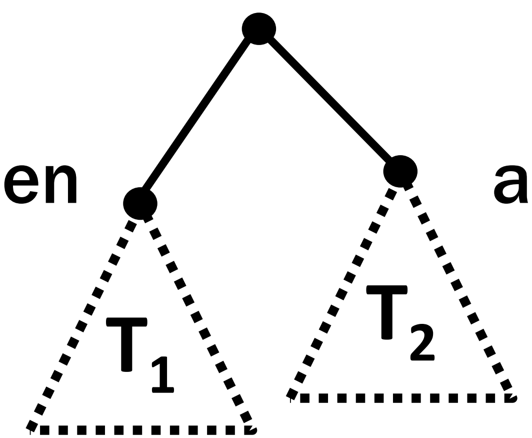
Rooted Binary Trees

- Basis:  is a rooted binary tree
- Recursive step:

If 

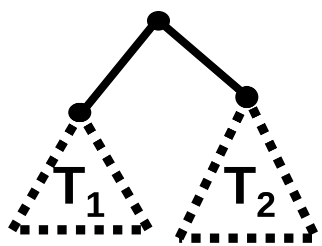
and 

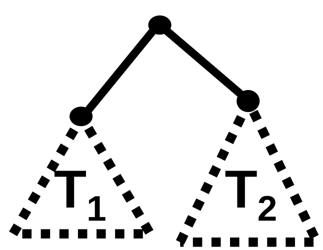
are rooted binary trees,

then 

also is a rooted binary tree.

Defining Functions on Rooted Binary Trees

- size(•) = 1
- size (

) = 1 + size($\mathbf{T_1}$) + size($\mathbf{T_2}$)
- height(•) = 0
- height (

)=1 + max{height($\mathbf{T_1}$), height($\mathbf{T_2}$)}

Structural Induction

How to prove $\forall\, x \in S, P(x)$ is true:

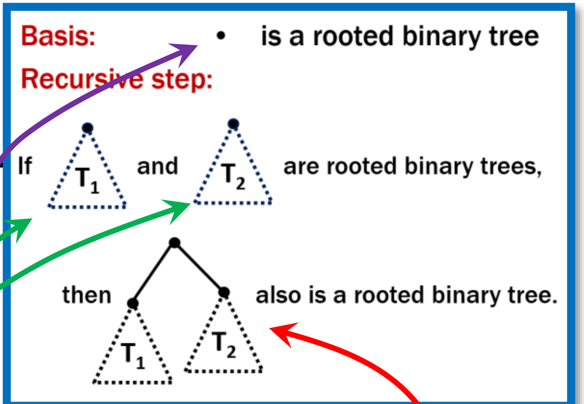
Base Case: **Show that $P(u)$ is true for all specific elements u of S mentioned in the *Basis step***

Inductive Hypothesis: **Assume that P is true for some arbitrary values of *each* of the existing named elements mentioned in the *Recursive step***

Inductive Step: **Prove that $P(w)$ holds for each of the new elements w constructed in the *Recursive step* using the named elements mentioned in the Inductive Hypothesis**

Conclude **that $\forall\, x \in S, P(x)$**

Structural Induction



How to prove $\forall\, x \in S, P(x)$ is true:

Base Case: Show that $P(u)$ is true for all **specific elements** u of S mentioned in the ***Basis step***

Inductive Hypothesis: Assume that P is true for some arbitrary values of *each* of the **existing named elements** mentioned in the ***Recursive step***

Inductive Step: Prove that $P(w)$ holds for each of the **new elements** w constructed in the ***Recursive step*** using the named elements mentioned in the Inductive Hypothesis

Conclude that $\forall\, x \in S, P(x)$

Structural Induction vs. Ordinary Induction

Ordinary induction is a special case of structural induction:

Recursive definition of $\mathbb{N}$

Basis: $0 \in \mathbb{N}$

Recursive step: If $k \in \mathbb{N}$ then $k + 1 \in \mathbb{N}$

Structural induction follows from ordinary induction:

Define $Q(n)$ to be "for all $x \in S$ that can be constructed in at most n recursive steps, $P(x)$ is true."

Using Structural Induction

- Let S be given by…
  - **Basis:** $6 \in S$; $15 \in S$;
  - **Recursive:** if $x, y \in S$ then $x + y \in S$.

Claim: Every element of S is divisible by 3.

Claim: Every element of S is divisible by 3.

1. Let $\mathsf{P}(x)$ be "$3|x$". We prove that $\mathsf{P}(x)$ is true for all $x \in S$ by structural induction.
2. Base Case: $3|6$ and $3|15$ so $\mathsf{P}(6)$ and $\mathsf{P}(15)$ are true

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: Every element of S is divisible by 3.

1. Let P(x) be "3|x". We prove that P(x) is true for all x ∈ S by structural induction.
2. Base Case: 3|6 and 3|15 so P(6) and P(15) are true
3. Inductive Hypothesis: Suppose that P(x) and P(y) are true for some arbitrary x,y ∈ S
4. Inductive Step: Goal: Show P(x+y)

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: Every element of S is divisible by 3.

1. Let P(x) be "3|x". We prove that P(x) is true for all x ∈ S by structural induction.
2. Base Case: 3|6 and 3|15 so P(6) and P(15) are true
3. Inductive Hypothesis: Suppose that P(x) and P(y) are true for some arbitrary x,y ∈ S
4. Inductive Step: Goal: Show P(x+y)

   Since P(x) is true, 3|x and so x=3m for some integer m and since P(y) is true, 3|y and so y=3n for some integer n. Therefore x+y=3m+3n=3(m+n) and thus 3|(x+y).

   Hence P(x+y) is true.
5. Therefore by induction 3|x for all x ∈ S.

Basis: $6 \in S$; $15 \in S$;

Recursive: if $x, y \in S$ then $x + y \in S$

Claim: $\mathrm{len}(x \bullet y) = \mathrm{len}(x) + \mathrm{len}(y)$ **for all** $x, y \in \Sigma^*$

Let $P(y)$ **be** "$\mathrm{len}(x \bullet y) = \mathrm{len}(x) + \mathrm{len}(y)$ for all $x \in \Sigma^*$ ".
We prove $P(y)$ **for all** $y \in \Sigma^*$ **by structural induction.**

Claim: len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x $\in \Sigma^*$" .
We prove P(y) **for all** y $\in \Sigma^*$ **by structural induction.**

Base Case: y= ε. **For any** x $\in \Sigma^*$, len(x• ε) = len(x) = len(x) + len(ε) **since** len(ε)=0. **Therefore** P(ε) **is true**

Claim: len(x•y) = len(x) + len(y) **for all** x,y $\in \Sigma^*$

Let P(y) **be** "len(x•y) = len(x) + len(y) for all x $\in \Sigma^*$" .
We prove P(y) **for all** y $\in \Sigma^*$ **by structural induction.**

Base Case: y= ε. **For any** x $\in \Sigma^*$, len(x• ε) = len(x) = len(x) + len(ε) **since** len(ε)=0. **Therefore** P(ε) **is true**

Inductive Hypothesis: Assume that P(w) **is true for some arbitrary** w $\in \Sigma^*$

Inductive Step: **Goal: Show that** P(wa) **is true for every** a $\in \Sigma$

Claim: $\mathrm{len}(x \bullet y) = \mathrm{len}(x) + \mathrm{len}(y)$ **for all** $x, y \in \Sigma^*$

Let $P(y)$ **be** "$\mathrm{len}(x \bullet y) = \mathrm{len}(x) + \mathrm{len}(y)$ for all $x \in \Sigma^*$" .
We prove $P(y)$ **for all** $y \in \Sigma^*$ **by structural induction.**

Base Case: $y = \varepsilon$. **For any** $x \in \Sigma^*$, $\mathrm{len}(x \bullet \varepsilon) = \mathrm{len}(x) = \mathrm{len}(x) + \mathrm{len}(\varepsilon)$ **since** $\mathrm{len}(\varepsilon)=0$. **Therefore** $P(\varepsilon)$ **is true**

Inductive Hypothesis: **Assume that** $P(w)$ **is true for some arbitrary** $w \in \Sigma^*$

Inductive Step: **Goal: Show that** $P(wa)$ **is true for every** $a \in \Sigma$

Let $a \in \Sigma$. **Let** $x \in \Sigma^*$. **Then** $\mathrm{len}(x \bullet wa) = \mathrm{len}((x \bullet w)a)$ **by defn of** $\bullet$
$= \mathrm{len}(x \bullet w)+1$ **by defn of len**
$= \mathrm{len}(x)+\mathrm{len}(w)+1$ **by I.H.**
$= \mathrm{len}(x)+\mathrm{len}(wa)$ **by defn of len**

Therefore $\mathrm{len}(x \bullet wa) = \mathrm{len}(x)+\mathrm{len}(wa)$ **for all** $x \in \Sigma^*$, **so** $P(wa)$ **is true.**

So, by induction $\mathrm{len}(x \bullet y) = \mathrm{len}(x) + \mathrm{len}(y)$ **for all** $x, y \in \Sigma^*$

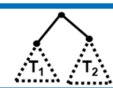
Claim: For every rooted binary tree **T**, $\text{size}(\mathbf{T}) \leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

Claim: For every rooted binary tree T, $\text{size}(\mathbf{T}) \leq 2^{\text{height}(\mathbf{T}) + 1} - 1$

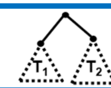
1. **Let** $P(\mathbf{T})$ **be** "$\text{size}(T) \leq 2^{\text{height}(T)+1} - 1$". **We prove** $P(\mathbf{T})$ **for all rooted binary trees T by structural induction.**
2. **Base Case:** $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ **and** $1=2^1-1=2^{0+1}-1$ **so** $P(\bullet)$ **is true.**

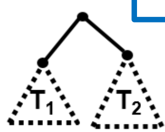
Claim: For every rooted binary tree T, size(T) ≤ $2^{\text{height}(T) + 1} - 1$

1. **Let** P(**T**) **be** "size(T) ≤ $2^{\text{height}(T)+1}-1$". **We prove** P(**T**) **for all rooted binary trees T by structural induction.**
2. **Base Case:** size(•)=1, height(•)=0 **and** $1=2^1-1=2^{0+1}-1$ **so** P(•) **is true.**
3. **Inductive Hypothesis: Suppose that** P(**T_1**) **and** P(**T_2**) **are true for some rooted binary trees T_1 and T_2.**
4. **Inductive Step:**

Goal: Prove P().

Claim: For every rooted binary tree T, $\text{size}(T) \le 2^{\text{height}(T)+1} - 1$

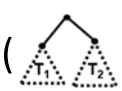
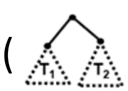
1. Let $P(T)$ be "$\text{size}(T) \le 2^{\text{height}(T)+1}-1$". We prove $P(T)$ for all rooted binary trees T by structural induction.
2. **Base Case:** $\text{size}(\bullet)=1$, $\text{height}(\bullet)=0$ and $1=2^1-1=2^{0+1}-1$ so $P(\bullet)$ is true.
3. **Inductive Hypothesis:** Suppose that $P(T_1)$ and $P(T_2)$ are true for some rooted binary trees T_1 and T_2.
4. **Inductive Step:** Goal: Prove $P($  $)$.

By defn, size( $) = 1+\text{size}(T_1)+\text{size}(T_2)$

$\le 1+2^{\text{height}(T_1)+1}-1+2^{\text{height}(T_2)+1}-1$ by IH for T_1 and T_2

$\le 2^{\text{height}(T_1)+1}+2^{\text{height}(T_2)+1}-1$

$\le 2(2^{\max(\text{height}(T_1),\text{height}(T_2))+1})-1$

$\le 2(2^{\text{height}($  $)})-1 \le 2^{\text{height}($  $)+1}-1$

which is what we wanted to show.

5. So, the $P(T)$ is true for all rooted bin. trees by structural induction.