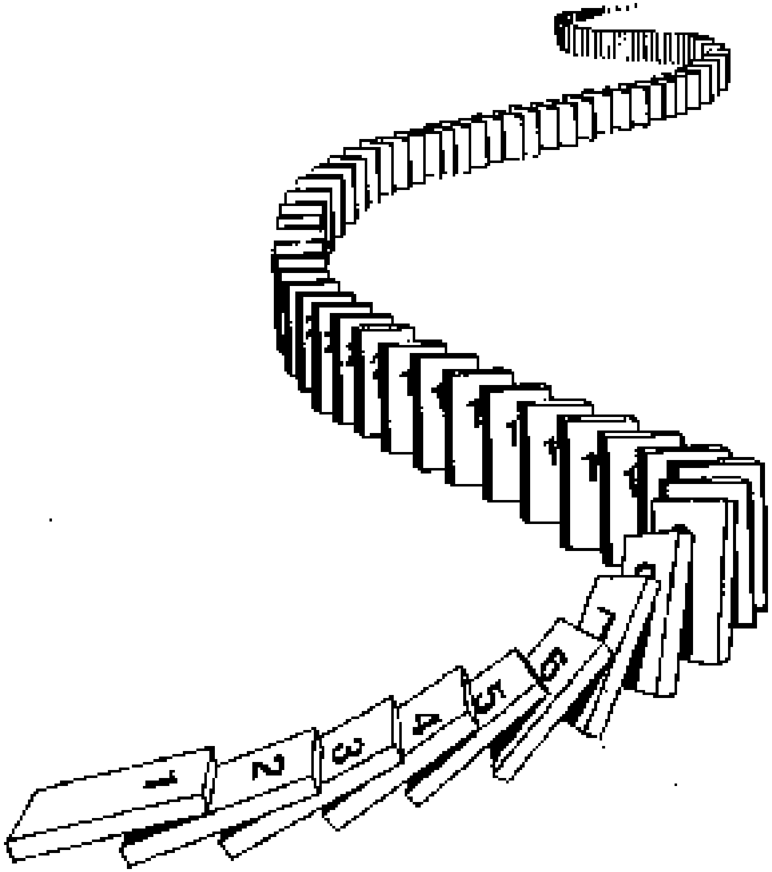
CSE 311: Foundations of Computing

Lecture 15: Induction & Strong Induction



Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq 0$ by induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq 0$"

Induction: Changing the start line

- What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b?

- Define predicate $Q(k) = P(k+b)$ for all k.
  - Then $\forall n\, Q(n) \equiv \forall n \geq b\;\; P(n)$

- Ordinary induction for Q:
  - Prove $Q(0) \equiv P(b)$
  - Prove

    $\forall k \left(Q(k) \longrightarrow Q(k+1)\right) \equiv \forall k \geq b \left(P(k) \longrightarrow P(k+1)\right)$

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq \boldsymbol{b}$ by induction."
2. "Base Case:" Prove $P(\boldsymbol{b})$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq \boldsymbol{b}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Inductive Proofs In 5 Easy Steps

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5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

$\cancel{3^{k+1} \geq (k+1)^2+3}$

1. Let $P(n)$ be "$3^n \geq n^2+3$". We prove $P(n)$ for $n \geq 2$ by induction.

2. Prove $P(2)$. $9 = 3^2 \geq 2^2 + 3 = 7$ ✓

3. IH: Assume $P(k)$ is true for some arbitrary $k \geq 2$.

4. Goal: $P(k+1)$ is true. $3^{k+1} \geq (k+1)^2+3 = k^2+2k+1+3$

   We know $3^k \geq k^2+3$ by IH.

   $$3^{k+1} = 3\cdot(3^k) \geq 3\cdot(k^2+3) = 3k^2+9 = k^2+k^2+k^2+9$$

   $k^2 \geq k$ because $k \geq 1$

   $$\geq k^2+k+k+4$$
   $$= (k+1)^2+3.$$

   $P(k+1)$ is true

5. $P(n)$ is true for all $n \geq 2$.

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2):

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$**.**
4. **Inductive Step:**

   Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** $n \geq 2$ **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 2$**.**
4. **Inductive Step:**

$$\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 3^{k+1} \geq (k+1)^2+3=k^2+2k+4}$$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let P(n) be "$3^n \geq n^2+3$". We will show P(n) is true for all integers $n \geq 2$ by induction.
2. Base Case (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ so P(2) is true.
3. Inductive Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 2$.
4. Inductive Step:

   $\boxed{\text{Goal: Show } P(k+1)\text{, i.e. show } 3^{k+1} \geq (k+1)^2+3=k^2+2k+4}$

   $$\begin{aligned} 3^{k+1} &= 3(3^k) \\ &\geq 3(k^2+3) \text{ by the IH} \\ &= k^2+2k^2+9 \\ &\geq k^2+2k+4 = (k+1)^2+3 \text{ since } k \geq 1. \end{aligned}$$

   Therefore P(k+1) is true.
5. Thus P(n) is true for all integers $n \geq 2$, by induction.

Midterm

- A week today (Monday, Nov 7) in class
- Closed book, closed notes
  - You will get lists of inference rules & equivalences
- Covers material up to end of ordinary induction.
- Practice problems & practice midterm on the website
  - Solutions later this week

- I will run a review session Sunday, Nov 6, 4:00-7:00 pm in EEB 105.

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

How do the givens prove P(5)?

$$\begin{tikzcd}
P(0) \arrow[r, bend left, "P(0)\rightarrow P(1)"] & P(1) \arrow[r, bend left, "P(1)\rightarrow P(2)"] & P(2) \arrow[r, bend left, "P(2)\rightarrow P(3)"] & P(3) \arrow[r, bend left, "P(3)\rightarrow P(4)"] & P(4) \arrow[r, bend left, "P(4)\rightarrow P(5)"] & P(5)
\end{tikzcd}$$

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$P(0) \xrightarrow{P(0)\to P(1)} P(1) \xrightarrow{P(1)\to P(2)} P(2) \xrightarrow{P(2)\to P(3)} P(3) \xrightarrow{P(3)\to P(4)} P(4) \xrightarrow{P(4)\to P(5)} P(5)$$

We made it harder than we needed to …

- **When we proved** $P(2)$ **we knew BOTH** $P(0)$ **and** $P(1)$
- **When we proved** $P(3)$ **we knew** $P(0)$ **and** $P(1)$ **and** $P(2)$
- **When we proved** $P(4)$ **we knew** $P(0)$, $P(1)$, $P(2)$, $P(3)$
- **etc.**

That's the essence of the idea of Strong Induction.

Strong Induction

$$P(0)$$

$$\forall k \ \Big(\big(P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)\big) \to P(k+1)\Big)$$

$$\therefore \forall n \, P(n)$$

Strong Induction

$$P(0)$$

$$\forall k\ \Big(\big(P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)\big) \to P(k+1)\Big)$$

$$\therefore\ \forall n\ P(n)$$

Strong induction for P follows from ordinary induction for Q where

$$Q(k) = P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k)$$

$Q(k) \to Q(k+1)$

Note that $Q(0) \equiv P(0)$ and $Q(k+1) \equiv Q(k) \ \wedge P(k+1)$ and $\forall n\ Q(n) \equiv \forall n\ P(n)$

Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq b$ by induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(k)$ is true"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)

5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ is true for every integer j from b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ are true) and point out where you are using it.
   (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

48 = 2 • 2 • 2 • 2 • 3
591 = 3 • 197
45,523 = 45,523
321,950 = 2 • 5 • 5 • 47 • 137
1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

We use strong induction to prove that a factorization into primes exists, but not that it is unique.

Every integer ≥ 2 is a product of primes.

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**

3. I.H. Assume for some arbitrary int $k \geq 2$, for all $2 \leq j \leq k$ $P(j)$ hold.

4. I.S Goal: $P(k+1)$ is true. $k+1$ is a product of primes.

Case 1: $k+1$ is a prime. Then it is a product of primes ✓

Case 2: $k+1$ is a composite. $k+1 = a \cdot b$ for integer $1 < a, b < k+1$

$2 \leq a, b \leq k$. $P(a)$, $P(b)$ hold by I.H.

$a = p_1 p_2 \cdots p_r$ $\quad b = q_1 q_2 \cdots q_s$ for primes $p_1, \ldots, p_r, q_1, \ldots, q_s$.

$k+1 = a \cdot b = p_1 \cdots p_r q_1 \cdots q_s$ ✓

Since $k+1 \geq 2$ no other cases. This implies $P(k+1)$.

5. We proved $P(n)$ for all $n \geq 2$.

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** k ≥ 2,
   P(j) **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
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   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{; i.e. } k+1 \textbf{ is a product of primes}}$

   <u>**Case:** k+1 **is prime**</u>**: Then by definition** k+1 **is a product of primes**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n $\geq$ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
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   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

   <u>**Case:** k+1 **is prime**</u>**: Then by definition** k+1 **is a product of primes**
   <u>**Case:** k+1 **is composite:**</u> **Then** k+1=ab **for some integers** a **and** b
   where 2 ≤ a, b ≤ k.

Every integer ≥ 2 is a product of primes.

1. **Let** $P(n)$ **be** "n is a product of primes". **We will show that** $P(n)$ **is true for all integers** $n \geq 2$ **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore $P(2)$ **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** $k \geq 2$, $P(j)$ **is true for every integer** j **between** 2 **and** k
4. **Inductive Step:**

   Goal: Show $P(k+1)$; **i.e.** $k+1$ **is a product of primes**

   <u>Case: $k+1$ is prime</u>: **Then by definition** $k+1$ **is a product of primes**
   <u>Case: $k+1$ is composite:</u> **Then** $k+1=ab$ **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** $P(a)$ **and** $P(b)$ **are true so we have**
   $a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$
   for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.
   Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

Every integer ≥ 2 is a product of primes.

1. **Let** P(n) **be** "n is a product of primes". **We will show that** P(n) **is true for all integers** n $\geq$ 2 **by strong induction.**
2. **Base Case** (n=2): 2 **is prime, so it is a product of primes.**
   Therefore P(2) **is true.**
3. **Inductive Hyp: Suppose that for some arbitrary integer** k $\geq$ 2,
   P(j) **is true for every integer** j **between** 2 **and** k $\forall_{2 \leq j \leq k} \cdot P(j) \to P(k+1)$
4. **Inductive Step:**

   Goal: Show P(k+1); **i.e.** k+1 **is a product of primes**

   <u>**Case:** k+1 **is prime**</u>: **Then by definition** k+1 **is a product of primes**

   <u>**Case:** k+1 **is composite:**</u> **Then** k+1=ab **for some integers** a **and** b **where** $2 \leq a, b \leq k$. **By our IH,** P(a) **and** P(b) **are true so we have**

   $a = p_1 p_2 \cdots p_r$ **and** $b = q_1 q_2 \cdots q_s$

   for some primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$.

   Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ **which is a product of primes.**

   Since k $\geq$ 1, **one of these cases must happen and so** P(k+1) **is true:**
5. **Thus** P(n) **is true for all integers** n $\geq$ 2, **by strong induction.**

Strong Induction is particularly useful when…

…we need to analyze methods that on input k make a recursive call for an input different from $k-1$.

e.g.: Recursive Modular Exponentiation:

– For exponent $k > 0$ it made a recursive call with exponent $\mathrm{j} = k/2$ when k was even or $\mathrm{j} = k-1$ when k was odd. e

$\le 2 \lg(k)$

We won't analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.

Recursive definitions of functions

- $F(0) = 0;\ \ F(n+1) = F(n) + 1$ **for all** $n \geq\ 0$**.**

  $F(n) = n$

- $G(0) = 1;\ \ G(n+1) = 2 \cdot G(n)$ **for all** $n \geq\ 0$**.**

  $G(n) = 2^n$

- $0! = 1;\ \ (n+1)! = (n+1) \cdot n!$ **for all** $n \geq\ 0$**.**

  $n! = n\,(n-1)\,(n-2) \cdots 1$

- $H(0) = 1;\ \ H(n+1) = 2^{H(n)}$ **for all** $n \geq\ 0$**.**

$H(0) = 1$

$H(1) = 2^1$

$H(2) = 2^{2^1} = 4$

$H(3) = 2^{2^{2^1}} = 16$

$H(4) = 2^{16} = 65000$

$H(n) = 2^{2^{2^{\cdot^{\cdot^{2}}}}} \Big\} n =$

$H(5) = 2^{65000}$

Prove $n! \leq n^n$ for all $n \geq 1$

1. Let $P(n)$ be "$n! \leq n^n$". We prove $P(n)$ for $n \geq 1$

2. Base case $P(1)$: $1 = 1! \leq 1^1 = 1$ ✓

3. IH: $P(k)$ holds for some $k \geq 1$

4. IS. Goal $P(k+1)$ $(k+1)! \leq (k+1)^{k+1} = (k+1).(k+1)^k$

by IH

$$(k+1)! = (k+1).\underbrace{k!} \leq (k+1).\underbrace{k^k}$$

$k \leq k+1$

$$\leq (k+1).(k+1)^k$$

$$= (k+1)^{k+1}$$

this gives $P(k+1)$

5. $P(n)$ holds $\forall\, n \geq 1$.

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "$n! \leq n^n$". **We will show that** P(n) **is true for all integers** $n \geq 1$ **by induction.**
2. **Base Case** (n=1): $1! = 1 \cdot 0! = 1 \cdot 1 = 1 = 1^1$ **so** P(1) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 1$.
4. **Inductive Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } (k+1)! \leq (k+1)^{k+1}}$

   $$\begin{aligned} (k+1)! &= (k+1) \cdot k! && \textbf{by definition of !} \\ &\leq (k+1) \cdot k^k && \textbf{by the IH} \\ &\leq (k+1) \cdot (k+1)^k && \textbf{since } k \geq 0 \\ &= (k+1)^{k+1} \end{aligned}$$

   Therefore P(k+1) **is true.**

5. **Thus** P(n) **is true for all** $n \geq 1$, **by induction.**