

Inductive Proofs In 5 Easy Steps

1. “Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq 0$ by induction.”
2. “Base Case:” Prove $P(0)$
3. “Inductive Hypothesis:
Assume $P(k)$ is true for some arbitrary integer $k \geq 0$ ”
4. “Inductive Step:” Prove that $P(k + 1)$ is true:
Use the goal to figure out what you need.
Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k + 1)$!!)
5. “Conclusion: $P(n)$ is true for all integers $n \geq 0$ ”

Induction: Changing the start line

- What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b ?
- Define predicate $Q(k) = P(k + b)$ for all k .
 - Then $\forall n Q(n) \equiv \forall n \geq b P(n)$
- Ordinary induction for Q :
 - Prove $Q(0) \equiv \underline{P(b)}$
 - Prove $\forall k (Q(k) \rightarrow Q(k + 1)) \equiv \underline{\forall k \geq b (P(k) \rightarrow P(k + 1))}$

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Assume $P(k)$ is true for some arbitrary integer $k \geq b$ ”

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5. “Conclusion: $P(n)$ is true for all integers $n \geq b$ ”

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5. “Conclusion: $P(n)$ is true for all integers $n \geq b$ ”

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let $P(n)$ be " $3^n \geq n^2 + 3$ ", we will prove that $P(n)$ is
-1

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let $P(n)$ be " $3^n \geq n^2 + 3$ ". We will show $P(n)$ is true for all integers $n \geq 2$ by induction.

2. Base Case ($n=2$): $3^2 = 9$ $2^2 + 3 = 4 + 3 = 7$
 $9 > 7$ so $P(2)$ is true.

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

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2. Base Case ($n=2$): $3^2 = 9 \geq 7 = 4 + 3 = 2^2 + 3$ so $P(2)$ is true.
3. *Inductive* Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 2$.
4. Induction Step:

Goal: Show $P(k+1)$, i.e. show $3^{k+1} \geq (k+1)^2 + 3$

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Goal: Show $P(k+1)$, i.e. show $3^{k+1} \geq (k+1)^2 + 3 = k^2 + 2k + 4$

$$3^{k+1} = 3(3^k)$$

$$\geq 3(k^2 + 3) \quad \text{by I.H.}$$

$$= 3k^2 + 9$$

$$= k^2 + 2k^2 + 9$$

$$\geq k^2 + 2k^2 + 4$$

$$\geq k^2 + 2k + 4$$

$$= \underline{(k+1)^2 + 3}$$

Since $k \geq 1$

$\therefore P(k+1)$ is true

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2. Base Case ($n=2$): $3^2 = 9 \geq 7 = 4 + 3 = 2^2 + 3$ so $P(2)$ is true.
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$$\begin{aligned} 3^{k+1} &= 3(3^k) \\ &\geq 3(k^2 + 3) \text{ by the IH} \\ &= k^2 + 2k^2 + 9 \\ &\geq k^2 + 2k + 4 = (k+1)^2 + 3 \text{ since } k \geq 1. \end{aligned}$$

Therefore $P(k+1)$ is true.

5. Thus $P(n)$ is true for all integers $n \geq 2$, by induction.

Midterm

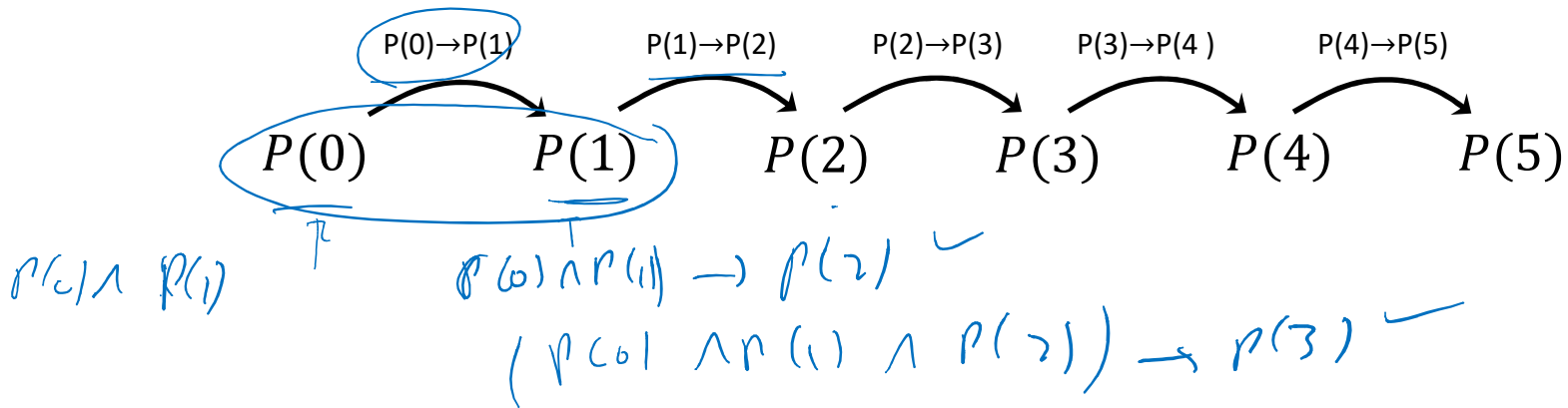
- **A week today (Monday, Nov 7) in class**
- **Closed book, closed notes**
 - You will get lists of inference rules & equivalences
- **Covers material up to end of ordinary induction.**
- **Practice problems & practice midterm on the website**
 - Solutions later this week
- **Prof. Oveis Gharan will run a review session Sunday, Nov 6, 4:00-7:00 pm in EEB 105.**

Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{l} P(0) \\ \forall k (P(k) \rightarrow P(k+1)) \\ \hline \therefore \forall n P(n) \end{array}$$

How do the givens prove P(5)?

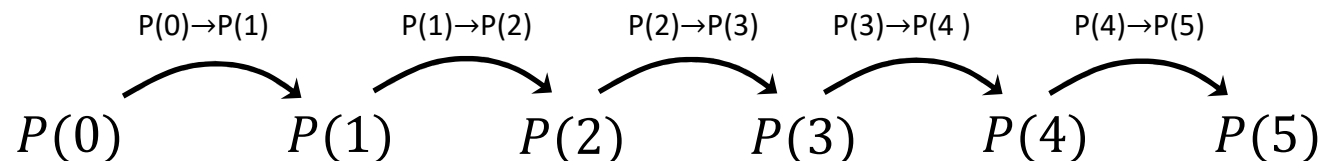


Recall: Induction Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \rightarrow \frac{\forall k (P(k) \rightarrow P(k + 1))}{\therefore \forall n P(n)} \end{array}$$

How do the givens prove $P(5)$?



We made it harder than we needed to ...

When we proved $P(2)$ we knew **BOTH** $P(0)$ and $P(1)$

When we proved $P(3)$ we knew $P(0)$ and $P(1)$ and $P(2)$

When we proved $P(4)$ we knew $P(0)$, $P(1)$, $P(2)$, $P(3)$

etc.

That's the essence of the idea of Strong Induction.

Strong Induction

$$P(0)$$

$$\forall k \left((P(0) \wedge P(1) \wedge P(2) \wedge \dots \wedge P(k)) \rightarrow P(k+1) \right)$$

$$\therefore \forall n P(n)$$

Strong induction for P follows from ordinary induction for Q where

$$Q(k) = P(0) \wedge P(1) \wedge P(2) \wedge \dots \wedge P(k)$$

Note that $Q(0) \equiv P(0)$ and $Q(k+1) \equiv Q(k) \wedge P(k+1)$ and $\forall n Q(n) \equiv \forall n P(n)$

$$(P(0) \wedge P(1) \wedge P(2)) \rightarrow P(3) \quad \text{E(1,2) \& } k=2 \cdot 2$$

$$P(3)$$

Inductive Proofs In 5 Easy Steps

1. “Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by induction.”
2. “Base Case:” Prove $P(b)$
3. “Inductive Hypothesis:
Assume that for some arbitrary integer $k \geq b$,
 $P(k)$ is true”
4. “Inductive Step:” Prove that $P(k + 1)$ is true:
Use the goal to figure out what you need.
Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k + 1)$!!)
5. “Conclusion: $P(n)$ is true for all integers $n \geq b$ ”

Strong Inductive Proofs In 5 Easy Steps

1. “Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by ***strong*** induction.”
2. “Base Case:” Prove $P(b)$
3. “Inductive Hypothesis:
Assume that for some arbitrary integer $k \geq b$,
 $P(j)$ is true for every integer j from b to k ”
4. “Inductive Step:” Prove that $P(k + 1)$ is true:
Use the goal to figure out what you need.
Make sure you are using I.H. (that $P(b), \dots, P(k)$ are true) and point out where you are using it.
(Don't assume $P(k + 1)$!!)
5. “Conclusion: $P(n)$ is true for all integers $n \geq b$ ”

Recall: Fundamental Theorem of Arithmetic

Every integer > 1 has a unique prime factorization

$$48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$591 = 3 \cdot 197$$

$$45,523 = 45,523$$

$$321,950 = 2 \cdot 5 \cdot 5 \cdot 47 \cdot 137$$

$$1,234,567,890 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 3,607 \cdot 3,803$$

We use strong induction to prove that a factorization into primes exists, but not that it is unique.

Every integer ≥ 2 is a product of primes.

1. Let $P(n)$ be "n is a product of primes". We will prove $P(n)$ for all integers $n \geq 2$

2. Base Case : $n=2$. 2 is prime so it is a product of primes by def.

Every integer ≥ 2 is a product of primes.

1. Let $P(n)$ be “ n is a product of primes”. We will show that $P(n)$ is true for all integers $n \geq 2$ by strong induction.
2. Base Case ($n=2$): 2 is prime, so it is a product of primes.
Therefore $P(2)$ is true.

Every integer ≥ 2 is a product of primes.

1. Let $P(n)$ be “ n is a product of primes”. We will show that $P(n)$ is true for all integers $n \geq 2$ by strong induction.
2. Base Case ($n=2$): 2 is prime, so it is a product of primes.
Therefore $P(2)$ is true.
3. Inductive Hyp: Suppose that for some arbitrary integer $k \geq 2$, $P(j)$ is true for every integer j between 2 and k
4. Inductive Step:

Goal: Show $P(k+1)$; i.e. $k+1$ is a product of primes

Case: $k+1$ is prime: It is a product of primes by defn ✓

Case: $k+1$ is composite

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2. Base Case ($n=2$): 2 is prime, so it is a product of primes.
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Goal: Show $P(k+1)$; i.e. $k+1$ is a product of primes

Case: $k+1$ is prime: Then by definition $k+1$ is a product of primes

Case: $k+1$ is composite: Then there are factors a, b s.t.
 $k+1 = a \cdot b$ and $1 < a, b < k+1$
i.e. $2 \leq a, b \leq k$

$\therefore P(a)$ and $P(b)$ are true by I.H.

$\therefore a, b$ are products of primes

so $k+1 = a \cdot b$ is a product of primes

\therefore Since $k+1 \geq 2$ one of these cases must hold. $\therefore P(k+1)$.

Every integer ≥ 2 is a product of primes.

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2. Base Case ($n=2$): 2 is prime, so it is a product of primes.
Therefore $P(2)$ is true.

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Goal: Show $P(k+1)$; i.e. $k+1$ is a product of primes

Case: $k+1$ is prime: Then by definition $k+1$ is a product of primes

Case: $k+1$ is composite: Then $k+1=ab$ for some integers a and b where $2 \leq a, b \leq k$.

Every integer ≥ 2 is a product of primes.

1. Let $P(n)$ be “ n is a product of primes”. We will show that $P(n)$ is true for all integers $n \geq 2$ by strong induction.
2. Base Case ($n=2$): 2 is prime, so it is a product of primes.
Therefore $P(2)$ is true.
3. Inductive Hyp: Suppose that for some arbitrary integer $k \geq 2$, $P(j)$ is true for every integer j between 2 and k
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Goal: Show $P(k+1)$; i.e. $k+1$ is a product of primes

Case: $k+1$ is prime: Then by definition $k+1$ is a product of primes

Case: $k+1$ is composite: Then $k+1=ab$ for some integers a and b where $2 \leq a, b \leq k$. By our IH, $P(a)$ and $P(b)$ are true so we have

$$a = p_1 p_2 \cdots p_r \text{ and } b = q_1 q_2 \cdots q_s$$

for some primes $p_1, p_2, \dots, p_r, q_1, q_2, \dots, q_s$.

Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ which is a product of primes.



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$$a = p_1 p_2 \cdots p_r \text{ and } b = q_1 q_2 \cdots q_s$$

for some primes $p_1, p_2, \dots, p_r, q_1, q_2, \dots, q_s$.

Thus, $k+1 = ab = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s$ which is a product of primes.

Since $k \geq 1$, one of these cases must happen and so $P(k+1)$ is true:

5. Thus $P(n)$ is true for all integers $n \geq 2$, by strong induction.

Strong Induction is particularly useful when...

...we need to analyze methods that on input k make a recursive call for an input different from $k - 1$.

e.g.: Recursive Modular Exponentiation:

- For exponent $k > 0$ it made a recursive call with exponent $j = k/2$ when k was even or $j = k - 1$ when k was odd.**

We won't analyze this particular method by strong induction, but we could.

However, we will use strong induction to analyze other functions with recursive definitions.

Recursive definitions of functions

- $F(0) = 0$; $F(n + 1) = F(n) + 1$ for all $n \geq 0$.

$$F(n) = n$$

- $G(0) = 1$; $G(n + 1) = 2 \cdot G(n)$ for all $n \geq 0$.

$$G(n) = 2^n$$

- $0! = 1$; $(n + 1)! = (n + 1) \cdot n!$ for all $n \geq 0$.

- $H(0) = 1$; $H(n + 1) = 2^{H(n)}$ for all $n \geq 0$.

$$H(n) = 2^{2^{2^{\dots^2}}}$$

Prove $n! \leq n^n$ for all $n \geq 1$

Prove $n! \leq n^n$ for all $n \geq 1$

1. Let $P(n)$ be " $n! \leq n^n$ ". We will show that $P(n)$ is true for all integers $n \geq 1$ by induction.
2. Base Case ($n=1$): $1! = 1 \cdot 0! = 1 \cdot 1 = 1 = 1^1$ so $P(1)$ is true. ✓
3. Inductive Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 1$.

4. Inductive Step:

Goal: Show $P(k+1)$, i.e. show $(k+1)! \leq (k+1)^{k+1}$

$$\begin{aligned}(k+1)! &= (k+1) \cdot k! && \text{by definition of !} \\ &\leq (k+1) \cdot k^k && \text{by the IH} \\ &\leq (k+1) \cdot (k+1)^k && \text{since } k \geq 0 \\ &= (k+1)^{k+1}\end{aligned}$$

Therefore $P(k+1)$ is true.

5. Thus $P(n)$ is true for all $n \geq 1$, by induction.