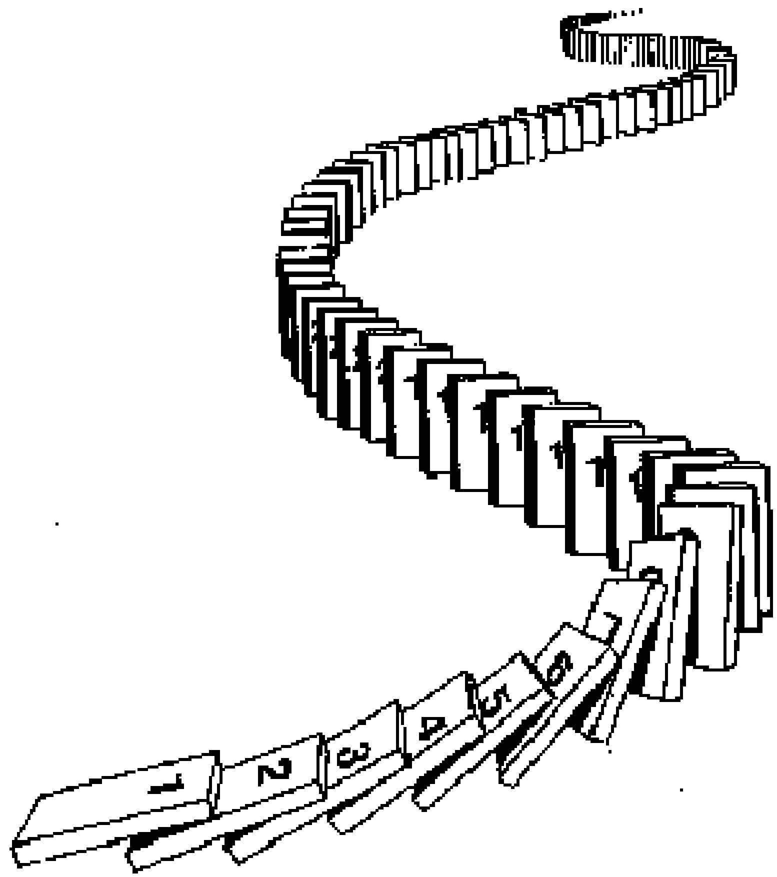
CSE 311: Foundations of Computing

Lecture 14: Induction

Look at bottom (right)
of course web page:

Handouts

Solving Modular Equations



Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily
- **Particularly useful for reasoning about programs!**

```
for(int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

```
public int f(int x) {
    if (x == 0) { return 0; }
    else { return f(x – 1); }
}
```

  - f(x) = x for all values of x ≥ 0 naturally shown by induction.

Prove $\forall a, b, m > 0\ \forall k \in \mathbb{N}\ (a \equiv b \pmod{m} \rightarrow a^k \equiv b^k \pmod{m})$

Let $a, b, m > 0 \in \mathbb{Z}$ be arbitrary. Let $k \in \mathbb{N}$ be arbitrary. Suppose that $a \equiv b \pmod{m}$.

We know $\big(a \equiv b \pmod{m} \wedge a \equiv b \pmod{m}\big) \rightarrow a^2 \equiv b^2 \pmod{m}$ by multiplying congruences. So, applying this repeatedly, we have:

$$\big(a \equiv b \pmod{m} \wedge a \equiv b \pmod{m}\big) \rightarrow a^2 \equiv b^2 \pmod{m}$$
$$\big(a^2 \equiv b^2 \pmod{m} \wedge a \equiv b \pmod{m}\big) \rightarrow a^3 \equiv b^3 \pmod{m}$$
$$\ldots$$
$$\left(a^{i-1} \equiv b^{i-1} \pmod{m} \wedge a \equiv b \pmod{m}\right) \rightarrow a^k \equiv b^k \pmod{m}$$

The "…"s is a problem! We don't have a proof rule that allows us to say "do this over and over".

But there such a property of the natural numbers!

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

How do the givens prove P(5)?

$\forall k\ (P(k) \rightarrow P(k+1))$

$P(0)$

$P(0) \rightarrow P(1)$ elim $\forall$

$P(1)$ MP

$P(1) \rightarrow P(2)$ elim $\forall$

$P(2)$ MP

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$P(0) \xrightarrow{P(0)\rightarrow P(1)} P(1) \xrightarrow{P(1)\rightarrow P(2)} P(2) \xrightarrow{P(2)\rightarrow P(3)} P(3) \xrightarrow{P(3)\rightarrow P(4)} P(4) \xrightarrow{P(4)\rightarrow P(5)} P(5)$$

First, we have P(0).

Since P(n) → P(n+1) for all n, we have P(0) → P(1).

Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.

Since P(n) → P(n+1) for all n, we have P(1) → P(2).

Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Using The Induction Rule In A Formal Proof

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

1. $P(0)$

4. $\forall k\,(P(k) \to P(k+1))$
5. $\forall n\ \ P(n)$ Induction from 1 & 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0

3. P(k) → P(k+1)
4. $\forall$k (P(k) $\rightarrow$ P(k+1)) Intro $\forall$ rule: 2, 3
5. $\forall$n P(n) Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0

   3.1 P(k) Assumption

   3.2 P(k+1) ?

3. P(k) → P(k+1) Direct Proof rule!
4. ∀k (P(k) → P(k+1)) Intro ∀: 2, 3
5. ∀n P(n) Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. Prove P(0)
2. Let k be an arbitrary integer ≥ 0
   - 3.1. Assume that P(k) is true
   - 3.2. ...
   - 3.3. Prove P(k+1) is true
3. $P(k) \rightarrow P(k+1)$ — Direct Proof Rule
4. $\forall k\ (P(k) \rightarrow P(k+1))$ — Intro $\forall$: 2, 3
5. $\forall n\ P(n)$ — Induction: 1, 4

Translating to an English Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

Domain: Integers ≥ 0
Natural #'s

in the domain

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0 — **Inductive Hypothesis**
    3.1. Assume that P(k) is true
    3.2. ... — **Inductive Step**
    3.3. Prove P(k+1) is true

3.	$P(k) \rightarrow P(k+1)$	Direct Proof Rule
4.	$\forall k\,(P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5.	$\forall n\, P(n)$	Induction: 1, 4

Conclusion

Translating To An English Proof

1. Prove P(0) — **Base Case**

2. Let k be an arbitrary integer ≥ 0
   3.1. Assume that P(k) is true — **Inductive Hypothesis**
   3.2. ...
   3.3. Prove P(k+1) is true — **Inductive Step**

3.	$P(k) \rightarrow P(k+1)$	Direct Proof Rule
4.	$\forall k\,(P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5.	$\forall n\, P(n)$	Induction: 1, 4

Conclusion

Induction Proof Template

[...Define P(n)...]

We will show that $P(n)$ is true for every $n \in \mathbb{N}$ by Induction.

Base Case: *[...proof of $P(0)$ here...]*

Induction Hypothesis:

arbitrary

Suppose that $P(k)$ is true for some $k \in \mathbb{N}$.

Induction Step:

We want to prove that $P(k+1)$ is true.

[...proof of $P(k+1)$ here...]

*The proof of $P(k+1)$ **must** invoke the IH somewhere.*

So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Proof:

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for some arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 \qquad = \quad 1$
- $1 + 2 \qquad = \quad 3$
- $1 + 2 + 4 \qquad = \quad 7$
- $1 + 2 + 4 + 8 \qquad = \quad 15$
- $1 + 2 + 4 + 8 + 16 \qquad = \quad 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.

Good that we have induction!

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

---

1. Let $P(n)$ be "$1+2+4+\ldots+2^n = 2^{n+1}-1$". We prove by induction that $P(n)$ is true for all $n \geq 0$.

don't use = here

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + 2^n = 2^{n+1} – 1". **We will show** P(n) **is true for all natural numbers by induction.**

2. Base case: n=0. P(0) says $1 = 2^{0+1} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $\underbrace{2^0}_{n=0} = 1 = 2 - 1 = \underbrace{2^{0+1} - 1}_{n=0}$ **so** P(0) **is true.** ✓

?

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$ $n \in \mathbb{N}$,

1. **Let** P(n) **be** "1 + 2 + … + $2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$.

4. Inductive Step: Goal: Show $P(k+1)$ i.e.
Show $1 + 2 + 4 + \ldots + 2^{k+1} = 2^{k+2} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

$\boxed{\textbf{Goal: Show } P(k+1) \textbf{, i.e. show } 1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1}$

By I.H. $\quad 1+2+4+\cdots+2^k = 2^{k+1} - 1$

Add 2^{k+1} to both sides

$$1+2+4+\cdots+2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$$

or $\quad 1+2+\cdots+2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$ by I.H.

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** $P(n)$ **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** $P(0)$ **is true.**
3. **Induction Hypothesis: Suppose that** $P(k)$ **is true for some arbitrary integer** $k \geq 0$**.**
4. **Induction Step:**

$$\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1}$$

$1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ **by IH**

Adding 2^{k+1} **to both sides, we get:**

$1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$**.**

So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$**, which is exactly** $P(k+1)$**.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** "$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** $P(n)$ **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** $P(0)$ **is true.**
3. **Induction Hypothesis: Suppose that** $P(k)$ **is true for some arbitrary integer** $k \geq 0$.
4. **Induction Step:**

   Goal: Show $P(k+1)$, **i.e. show** $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

   $1 + 2 + \ldots + 2^k = 2^{k+1} - 1$ **by IH**

   Adding 2^{k+1} **to both sides, we get:**

   $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $1 + 2 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, **which is exactly** $P(k+1)$.
5. **Thus** $P(n)$ **is true for all** $n \in \mathbb{N}$, **by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 3 + \dots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**

Prove $1 + 2 + 3 + \dots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**.**
4. **Induction Step:**

Goal: Show P(k+1), **i.e. show** $1 + 2 + \dots + \overset{k}{\cancel{n}} + (\overset{k}{\cancel{n}}+1) = (\overset{k}{\cancel{n}}+1)(\overset{k}{\cancel{n}}+2)/2$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0.
4. **Induction Step:**

   $\boxed{\textbf{Goal: Show } P(k+1)\textbf{, i.e. show } 1 + 2 + \ldots + n + (n+1) = (n+1)(n+2)/2}$

   $$1 + 2 + \ldots + \underset{k}{\cancel{n}} = \underset{k}{\cancel{n}}(\underset{k}{\cancel{n}}+1)/2 \quad \textbf{by IH}$$

   Adding n+1 **to both sides, we get:**

   $$1 + 2 + \ldots + \overset{k}{\cancel{n}} + (\overset{k}{\cancel{n}}+1) = \overset{k}{\cancel{n}}(\overset{k}{\cancel{n}}+1)/2 + (\overset{k}{\cancel{n}}+1)$$

   Now $\overset{k}{\cancel{n}}(\overset{k}{\cancel{n}}+1)/2 + (\overset{k}{\cancel{n}}+1) = (\overset{k}{\cancel{n}}+1)(\overset{k}{\cancel{n}}/2 + 1) = (\overset{k}{\cancel{n}}+1)(\overset{k}{\cancel{n}}+2)/2$.

   So, we have $1 + 2 + \ldots + \overset{k}{\cancel{n}} + (\overset{k}{\cancel{n}}+1) = (\overset{k}{\cancel{n}}+1)(\overset{k}{\cancel{n}}+2)/2$, **which is exactly** P(k+1).

5. **Thus** P(n) **is true for all** n ∈ℕ, **by induction.** $\forall n\, P(n)$
   $\therefore P(500)$

Another example of a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- …

Prove: $3 \mid (2^{2n} - 1)$ **for all** $n \geq 0$

Prove: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

1. **Let** P(n) **be** "3 | $(2^{2n}$ - 1$)$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0):

Prove: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

1. **Let** P(n) **be** "3 | 2^{2n} - 1". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}$-1=1-1=0=3·0 **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**.**
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** 3 | $(2^{2(k+1)} - 1)$

Prove: $3 \mid 2^{2n} - 1$ for all $n \geq 0$

1. **Let** P(n) **be** "3 | 2^{2n} - 1". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}-1=1-1=0=3\cdot 0$ **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**.**
4. **Induction Step:**

   Goal: Show P(k+1)**, i.e. show** $3 \mid (2^{2(k+1)} - 1)$

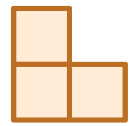
   By IH, $3 \mid (2^{2k} - 1)$ **so** $2^{2k} - 1 = 3j$ **for some integer** j

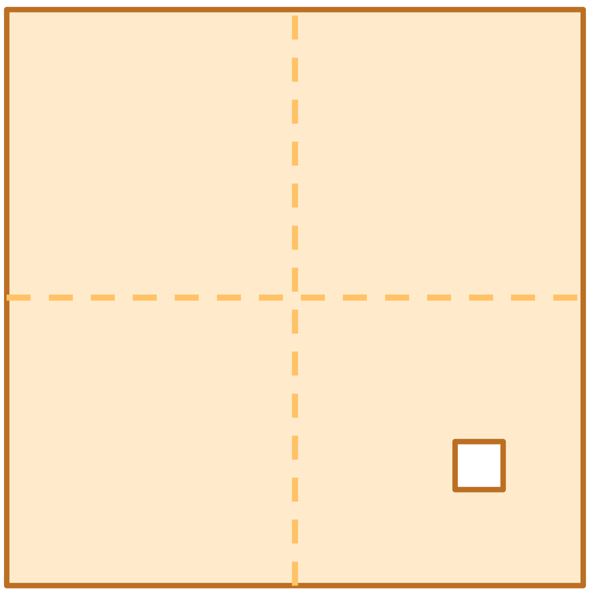
   So $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 4(2^{2k}) - 1 = 4(3j+1) - 1$

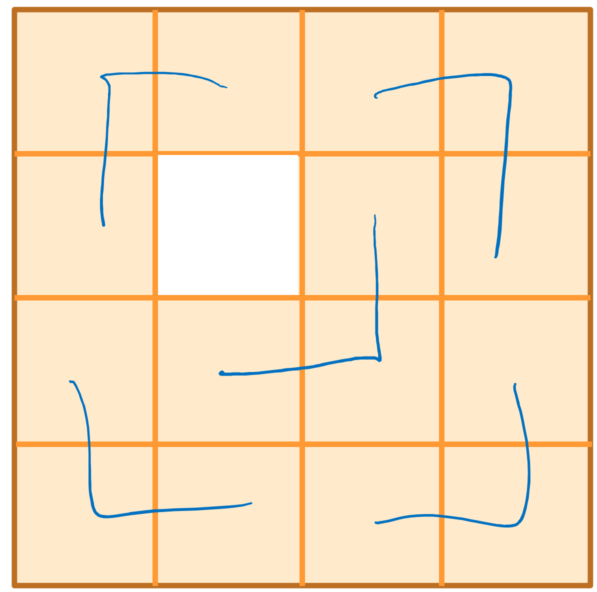
   $= 12j+3 = 3(4j+1)$

   Therefore $3 \mid (2^{2(k+1)} - 1)$ **which is exactly** P(k+1)**.**
5. **Thus** P(n) **is true for all** n∈ℕ**, by induction.**

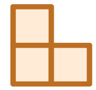
Checkerboard Tiling

- Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 

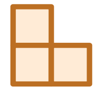


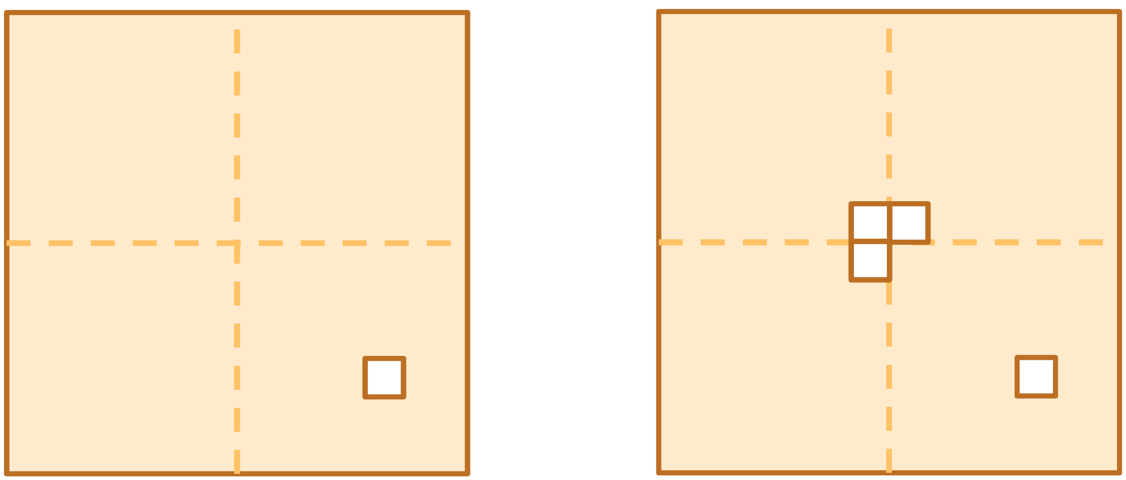


Checkerboard Tiling

1. **Let** $P(n)$ **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove $P(n)$ **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** $n=1$ 

Checkerboard Tiling

1. **Let** $P(n)$ **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove $P(n)$ **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** $n=1$ 
3. **Inductive Hypothesis: Assume** $P(k)$ **for some arbitrary integer** $k \geq 1$
4. **Inductive Step: Prove** $P(k+1)$



Apply IH to each quadrant then fill with extra tile.