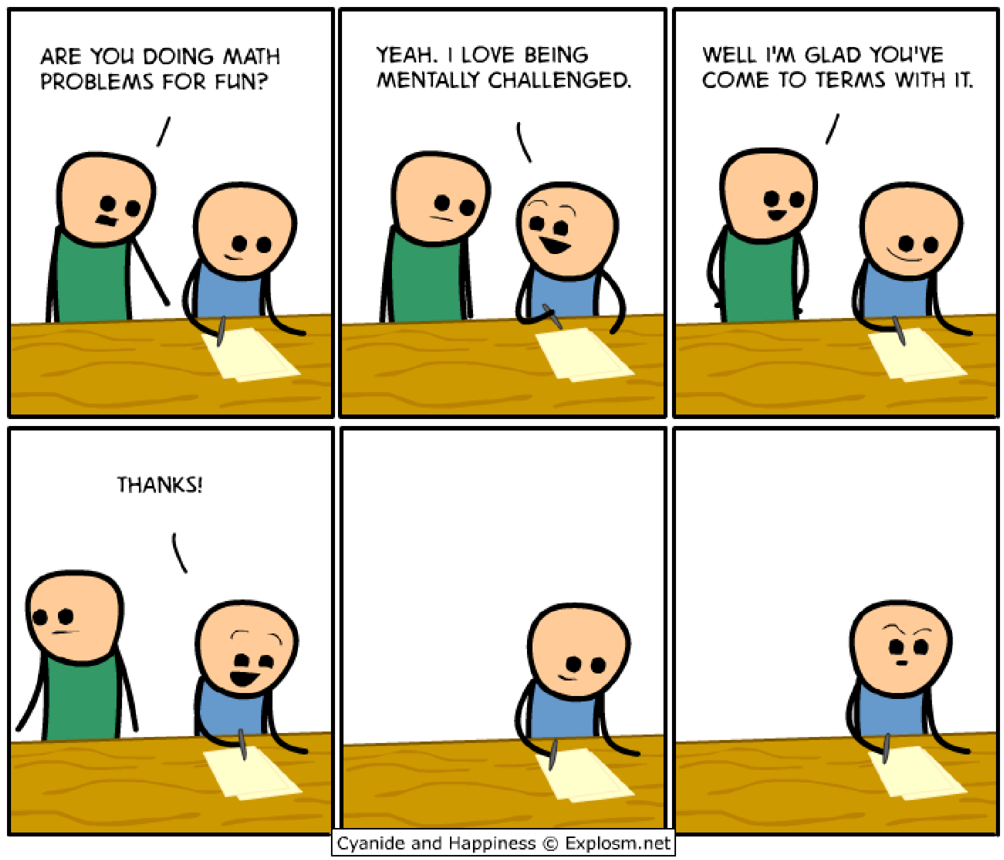
CSE 311: Foundations of Computing

Lecture 13: Modular Inverse, Exponentiation



Last time: Euclid's Algorithm

gcd(a, b) = gcd(b, a mod b), gcd(a,0)=a.

```
int gcd(int a, int b){ /* a >= b, b >= 0 */
    if (b == 0) {
        return a;
    }
    else {
        return gcd(b, a % b);
    }
```

Last time: Bézout's theorem

If a and b are positive integers, then there exist integers $\boldsymbol{s}$ and $\boldsymbol{t}$ such that

$$\gcd(a,b) = \boldsymbol{s}a + \boldsymbol{t}b.$$

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 3 (Backward Substitute Equations):

$8 = 35 - 1 * 27$

$3 = 27 - 3 * 8$

$2 = 8 - 2 * 3$

$1 = 3 - 1 * 2$

Plug in the def of 2

$1 = 3 - 1 * (8 - 2 * 3)$

$= 3 - 8 + 2 * 3$

$= (-1) * 8 + 3 * 3$

Re-arrange into 3's and 8's

Plug in the def of 3

$= (-1) * 8 + 3 * (27 - 3 * 8)$

$= (-1) * 8 + 3 * 27 + (-9) * 8$

$= 3 * 27 + (-10) * 8$

Re-arrange into 8's and 27's

$= 3 * 27 + (-10) * (35 - 1 * 27)$

$= 3 * 27 + (-10) * 35 + 10 * 27$

$= 13 * 27 + (-10) * 35$

Re-arrange into 27's and 35's

multiplicative inverse $\bmod m$

Suppose $\mathrm{GCD}(a, m) = 1$

By Bézout's Theorem, there exist integers s and t such that $sa + tm = 1$.

$s \bmod m$ is the multiplicative inverse of a:

$$1 = (sa + tm) \bmod m = sa \bmod m$$

Example

Solve: $7x \equiv 1 \pmod{26}$

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$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = 1$$

$$26 = 3*7 + 5 \qquad 5 = 26 - 3*7$$
$$7 = 1*5 + 2 \qquad 2 = 7 - 1*5$$
$$5 = 2*2 + 1 \qquad 1 = 5 - 2*2$$

$$\begin{aligned}1 &= 5 - 2*(7 - 1*5)\\ &= (-2)*7 + 3*5\\ &= (-2)*7 + 3*(26 - 3*7)\\ &= (-11)*7 + 3*26\end{aligned}$$

Multiplicative inverse of 7 mod 26

Now $(-11) \bmod 26 = 15$. **So,** $x = 15 + 26k$ **for** $k \in \mathbb{Z}$.

Example of a more general equation

Now solve: $7y \equiv 3 \pmod{26}$

We already computed that 15 is the multiplicative inverse of 7 modulo 26:

That is, $7 \cdot 15 \equiv 1 \pmod{26}$

By the multiplicative property of mod we have

$$7 \cdot 15 \cdot 3 \equiv 3 \pmod{26}$$

So any $y \equiv 15 \cdot 3 \pmod{26}$ is a solution.

That is, $y = 19 + 26k$ for any integer k is a solution.

Math mod a prime is especially nice

$\gcd(a, m) = 1$ **if** m **is prime and** $0 < a < m$ **so can always solve these equations mod a prime.**

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

X	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

mod 7

Modular Exponentiation mod 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2						
3						
4						
5						
6						

Exponentiation

- **Compute** 78365^{81453}
- **Compute** $78365^{81453} \bmod 104729$
- **Output is small**
  - – need to keep intermediate results small

Repeated Squaring – small and fast

Since $ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$

we have $a^2 \bmod m = (a \bmod m)^2 \bmod m$

and $a^4 \bmod m = (a^2 \bmod m)^2 \bmod m$

and $a^8 \bmod m = (a^4 \bmod m)^2 \bmod m$

and $a^{16} \bmod m = (a^8 \bmod m)^2 \bmod m$

and $a^{32} \bmod m = (a^{16} \bmod m)^2 \bmod m$

Can compute $a^k \bmod m$ for $k = 2^i$ in only i steps

What if k is not a power of 2?

Fast Exponentiation: $a^k \bmod m$ for all k

$$a^{2j} \bmod m = \left(a^j \bmod m\right)^2 \bmod m$$

$$a^{2j+1} \bmod m \ = \ \left((a \bmod m) \cdot (a^{2j} \bmod m)\right) \bmod m$$

Fast Exponentiation

```
public static long FastModExp(long a, long k, long modulus) {
        long result = 1;
        long temp;

        if (k > 0) {
            if ((k % 2) == 0) {
                temp = FastModExp(a,k/2,modulus);
                result = (temp * temp) % modulus;
            }
            else {
                temp = FastModExp(a,k-1,modulus);
                result = (a * temp) % modulus;
            }
        }
        return result;
    }
```

$$a^{2j} \bmod m = \left(a^j \bmod m\right)^2 \bmod m$$

$$a^{2j+1} \bmod m = \left((a \bmod m) \cdot (a^{2j} \bmod m)\right) \bmod m$$

Fast Exponentiation Algorithm

Another way: 81453 in binary is 10011111000101101

$81453 = 2^{16} + 2^{13} + 2^{12} + 2^{11} + 2^{10} + 2^{9} + 2^{5} + 2^{3} + 2^{2} + 2^{0}$

$a^{81453} = a^{2^{16}} \cdot a^{2^{13}} \cdot a^{2^{12}} \cdot a^{2^{11}} \cdot a^{2^{10}} \cdot a^{2^{9}} \cdot a^{2^{5}} \cdot a^{2^{3}} \cdot a^{2^{2}} \cdot a^{2^{0}}$

a^{81453} mod m=

(…(((((a$^{2^{16}}$ mod m ·

a$^{2^{13}}$ mod m) mod m ·

a$^{2^{12}}$ mod m) mod m ·

a$^{2^{11}}$ mod m) mod m ·

a$^{2^{10}}$ mod m) mod m ·

a$^{2^{9}}$ mod m) mod m ·

a$^{2^{5}}$ mod m) mod m ·

a$^{2^{3}}$ mod m) mod m ·

a$^{2^{2}}$ mod m) mod m ·

a$^{2^{0}}$ mod m) mod m

The fast exponentiation algorithm computes $a^k \bmod m$ using $O(\log k)$ multiplications $\bmod m$