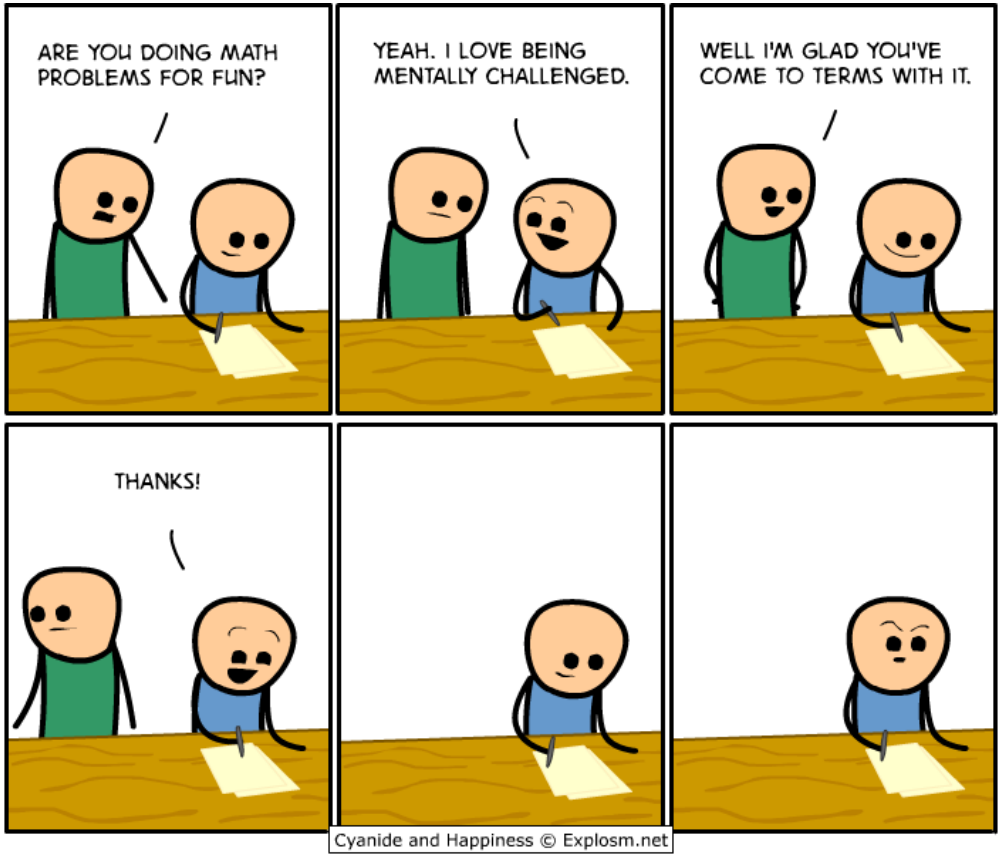
CSE 311: Foundations of Computing

Lecture 13: Modular Inverse, Exponentiation

Today starting at 2:00
We have asked folks from the Center for Engineering Teaching and Learning (CELT) to get your feedback on the class.
I will leave then but have my normal after-class office hours.



Last time: Euclid's Algorithm

gcd(a, b) = gcd(b, a mod b), gcd(a,0)=a.

```
int gcd(int a, int b){ /* a >= b, b >= 0 */
    if (b == 0) {
        return a;
    }
    else {
        return gcd(b, a % b);
    }
```

Last time: Bézout's theorem

If a and b are positive integers, then there exist integers $\boldsymbol{s}$ and $\boldsymbol{t}$ such that

$$\gcd(a,b) = \boldsymbol{s}a + \boldsymbol{t}b.$$

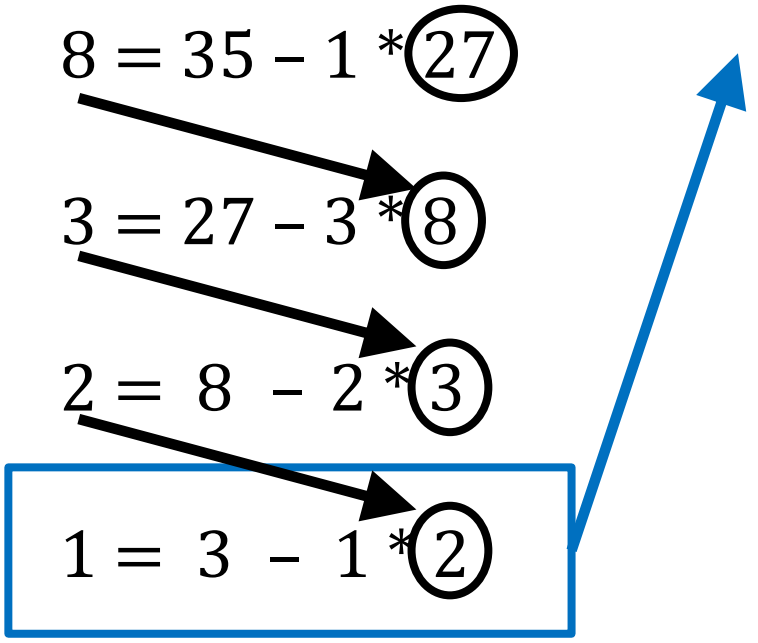
Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

gcd(35, 27)

Step 3 (Backward Substitute Equations):



Plug in the def of 2

$$\begin{aligned} 1 &= 3 - 1 * (8 - 2 * 3) \\ &= 3 - 8 + 2 * 3 \\ &= (-1) * 8 + 3 * 3 \end{aligned}$$

Re-arrange into 3's and 8's

Plug in the def of 3

$$\begin{aligned} &= (-1) * 8 + 3 * (27 - 3 * 8) \\ &= (-1) * 8 + 3 * 27 + (-9) * 8 \\ &= 3 * 27 + (-10) * 8 \end{aligned}$$

Re-arrange into 8's and 27's

$$\begin{aligned} &= 3 * 27 + (-10) * (35 - 1 * 27) \\ &= 3 * 27 + (-10) * 35 + 10 * 27 \\ &= 13 * 27 + (-10) * 35 \end{aligned}$$

Re-arrange into 27's and 35's

s t

multiplicative inverse $\bmod m$

Suppose $\mathrm{GCD}(a, m) = 1$

s, t

$sa + tm = 1$

$1 \bmod m = 1$

$(sa + tm) \bmod m = sa \bmod m$

By Bézout's Theorem, there exist integers s **and** t **such that** $sa + tm = 1$.

$s \bmod m$ **is the multiplicative inverse of** a**:**

$$1 = (sa + tm) \bmod m = sa \bmod m$$

$sa \equiv 1 \mod m$

$s = \frac{1}{a}$

Example

Solve: $7x \equiv 1 \pmod{26}$

gcd (7, 26).

$26 = 3 \cdot 7 + 5$.

$7 = 1 \cdot 5 + 2$

$5 = 2 \cdot 2 + 1$

$5 = 26 - 3 \cdot 7$

$2 = 7 - 1 \cdot 5$

$1 = 5 - 2 \cdot 2$

Example

$7 \cdot 15 = 105 = 4 \cdot 26 + 1$

$105 \bmod 26 = 1$

$7 \cdot 15 \equiv 1 \bmod 26$

Solve: $7x \equiv 1 \pmod{26}$

$$\gcd(26, 7) = \gcd(7, 5) = \gcd(5, 2) = \gcd(2, 1) = 1$$

$$\begin{aligned} 26 &= 3 * 7 + 5 & \quad 5 &= 26 - 3 * 7 \\ 7 &= 1 * 5 + 2 & \quad 2 &= 7 - 1 * 5 \\ 5 &= 2 * 2 + 1 & \quad 1 &= 5 - 2 * 2 \end{aligned}$$

$$\begin{aligned} 1 &= 5 - 2 * (7 - 1 * 5) \\ &= (-2) * 7 + 3 * 5 \\ &= (-2) * 7 + 3 * (26 - 3 * 7) \\ &= (-11) * 7 + 3 * 26 \end{aligned}$$

Multiplicative inverse of 7 mod 26

Now $(-11) \bmod 26 = 15$. **So,** $x = 15 + 26k$ **for** $k \in \mathbb{Z}$.

Example of a more general equation

Now solve: $7y \equiv 3 \pmod{26}$

$7_x \equiv 1 \mod 26$ ✓

$3 \equiv 3 \mod 26$

$7 3 . x \equiv 3 \mod 26$

We already computed that 15 is the multiplicative inverse of 7 modulo 26:

$7z = 21 \mod 26$

$15.7 = 60 \equiv 8 \mod 26$

$8 + 26k$.

That is, $7 \cdot 15 \equiv 1 \pmod{26}$

By the multiplicative property of mod we have

$$7 \cdot 15 \cdot 3 \equiv 3 \pmod{26}$$

$15.3 = 45 \equiv 19 \mod 26$

So any $y \equiv 15 \cdot 3 \pmod{26}$ is a solution.

That is, $y = 19 + 26k$ for any integer k is a solution.

Math mod a prime is especially nice

$a x \equiv b \mod 7$

$b \cdot a \cdot x \equiv b \mod 7$

$\gcd(a, m) = 1$ **if** m **is prime and** $0 < a < m$ **so can always solve these equations mod a prime.**

$a x \equiv b \mod m$

$0 < b < m$

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

$4x \equiv 5 \mod 7$

mod 7

Modular Exponentiation mod 7

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

→

a	a^1	a^2	a^3	a^4	a^5	a^6
1						
2	2	4	1	2	4	1
3						
4						
5	5	4	6	2	3	1
6						

Exponentiation

- **Compute** 78365^{81453}

400,000

- **Compute** $78365^{81453} \bmod 104729$

- **Output is small**
  - need to keep intermediate results small

Repeated Squaring – small and fast

Since $ab \bmod m = ((a \bmod m)(b \bmod m)) \bmod m$

we have $a^2 \bmod m = (a \bmod m)^2 \bmod m$

and $a^4 \bmod m = (a^2 \bmod m)^2 \bmod m$

and $a^8 \bmod m = (a^4 \bmod m)^2 \bmod m$

and $a^{16} \bmod m = (a^8 \bmod m)^2 \bmod m$

and $a^{32} \bmod m = (a^{16} \bmod m)^2 \bmod m$

Can compute $a^k \bmod m$ for $k = 2^i$ in only i steps

What if k is not a power of 2?

Fast Exponentiation: $a^k \bmod m$ for all k

$$\begin{cases} a^{2j} \bmod m = \left(a^j \bmod m\right)^2 \bmod m \\ a^{2j+1} \bmod m = \left((a \bmod m) \cdot (a^{2j} \bmod m)\right) \bmod m \end{cases}$$

Fast Exponentiation

```
public static long FastModExp(long a, long k, long modulus) {
        long result = 1;
        long temp;

        if (k > 0) {
            if ((k % 2) == 0) {
                temp = FastModExp(a,k/2,modulus);
                result = (temp * temp) % modulus;
            }
            else {
                temp = FastModExp(a,k-1,modulus);
                result = (a * temp) % modulus;
            }
        }
        return result;
    }
```

$a < \text{modulus}$

$a^{x+y} = a^x \cdot a^y$

$a^{x+y} \bmod m = (a^x \bmod m \cdot a^y \bmod m) \bmod m$

even

$$a^{2j} \bmod m = \left(a^j \bmod m\right)^2 \bmod m$$

$$a^{2j+1} \bmod m = \left((a \bmod m) \cdot (a^{2j} \bmod m)\right) \bmod m$$

Fast Exponentiation Algorithm

Another way: 81453 in binary is 1001111000101101

$$81453 = 2^{16} + 2^{13} + 2^{12} + 2^{11} + 2^{10} + 2^{9} + 2^{5} + 2^{3} + 2^{2} + 2^{0}$$

$$a^{81453} = a^{2^{16}} \cdot a^{2^{13}} \cdot a^{2^{12}} \cdot a^{2^{11}} \cdot a^{2^{10}} \cdot a^{2^{9}} \cdot a^{2^{5}} \cdot a^{2^{3}} \cdot a^{2^{2}} \cdot a^{2^{0}}$$

$a^{81453} \bmod m =$
$(\ldots(((((a^{2^{16}} \bmod m \cdot$
$a^{2^{13}} \bmod m) \bmod m \cdot$
$a^{2^{12}} \bmod m) \bmod m \cdot$
$a^{2^{11}} \bmod m) \bmod m \cdot$
$a^{2^{10}} \bmod m) \bmod m \cdot$
$a^{2^{9}} \bmod m) \bmod m \cdot$
$a^{2^{5}} \bmod m) \bmod m \cdot$
$a^{2^{3}} \bmod m) \bmod m \cdot$
$a^{2^{2}} \bmod m) \bmod m \cdot$
$a^{2^{0}} \bmod m) \bmod m$

The fast exponentiation algorithm computes
$a^k \bmod m$ **using** $\leq 2\log k$ **multiplications** $\bmod m$