# CSE 311: Foundations of Computing

## Lecture 12: Two's Complement, Primes, GCD

HW 3 #5:

checking your sol$^{n}$

correct sol$^{ns}$

& feedback.

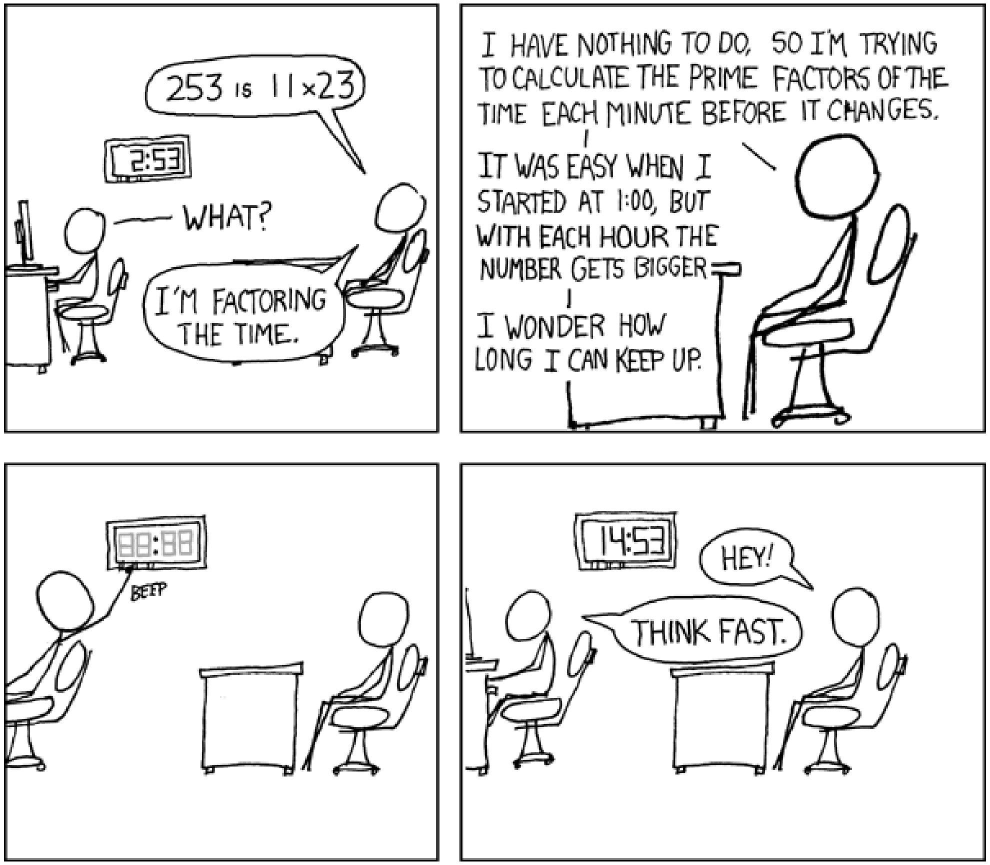
last time:

mod function

mod predicate

addition

multiplication



# n-bit Unsigned Integer Representation

- Represent integer $x$ as sum of powers of $2$:

  If $\sum_{i=0}^{n-1} b_i 2^i$ where each $b_i \in \{0,1\}$

  then representation is $b_{n-1}\ldots b_2\, b_1\, b_0$

99 = 64 + 32 + 2 + 1

18 = 16 + 2

- For n = 8:

  99: 0110 0011

  \+ 18: 0001 0010

  = 0 1 1 1 0 1 0 1 117

# Sign-Magnitude Integer Representation

$n$-bit signed integers

Suppose that $-2^{n-1} < x < 2^{n-1}$

First bit as the sign, $n-1$ bits for the value

99 = 64 + 32 + 2 + 1

18 = 16 + 2

For n = 8:

99:    0110  0011

-18:   1001  0010

-117    1111 0101

Any problems with this representation?    -0 and 0

# Two's Complement Representation

$n$ bit signed integers, first bit will still be the sign bit

Suppose that $0 \le x < 2^{n-1}$ ,
    $x$ is represented by the binary representation of $x$

same as before

Suppose that $0 < x \le 2^{n-1}$ ,
    $-x$ is represented by the binary representation of $2^n - x$

**Key property:** Twos complement representation of any number $\boldsymbol{y}$ is equivalent to $\boldsymbol{y}$ $\mathbf{mod}\ \mathbf{2^n}$ so arithmetic works $\mathbf{mod}\ \mathbf{2^n}$

99 = 64 + 32 + 2 + 1
18 = 16 + 2

For n = 8:
$n-1$

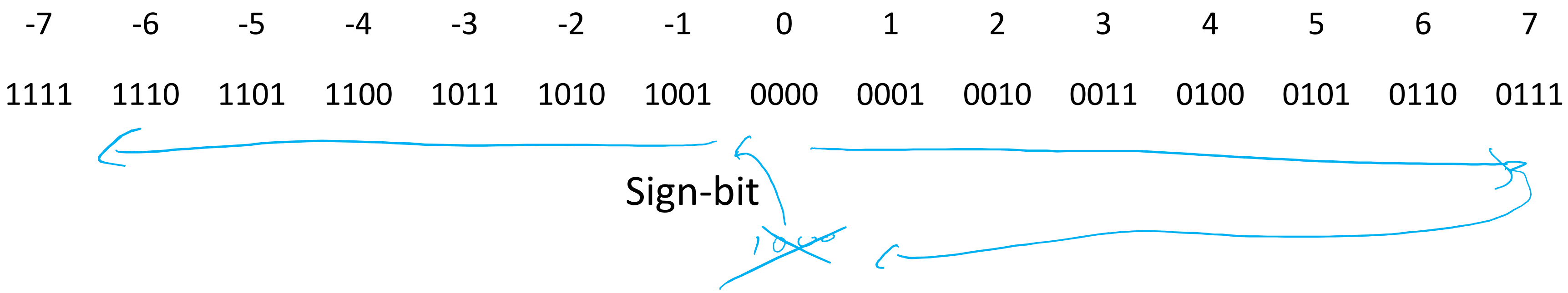
```
   99:   0110 0011
+ -18:   1110 1110
         ---------
         0101 0001
```

$64 + 16 + 1 = 81$

$$\begin{array}{r} \overset{1}{2}\overset{4}{\cancel{5}}\overset{1}{6} \\ -\ 18 \\ \hline 238 \end{array}$$

```
  11111111
  00010010
  --------
→ 11101101
         1
  --------
  11101110
```

# Sign-Magnitude vs. Two's Complement

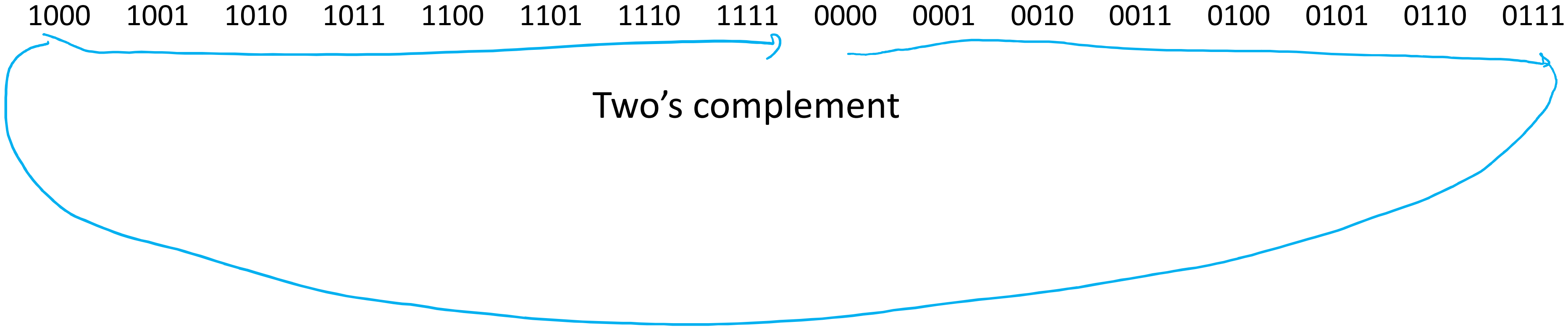


| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1111 | 1110 | 1101 | 1100 | 1011 | 1010 | 1001 | 0000 | 0001 | 0010 | 0011 | 0100 | 0101 | 0110 | 0111 |

Sign-bit

~~1000~~

$2^4$



| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1000 | 1001 | 1010 | 1011 | 1100 | 1101 | 1110 | 1111 | 0000 | 0001 | 0010 | 0011 | 0100 | 0101 | 0110 | 0111 |

Two's complement

# Two's Complement Representation

- For $0 < x \leq 2^{n-1}$, $-x$ is represented by the binary representation of $\underline{2^n - x}$
  - That is, the two's complement representation of any number $y$ has the same value as $y$ modulo $2^n$.

- To compute this: Flip the bits of $x$ then add 1:
  - All 1's string is $\underline{2^n - 1}$, so

    Flip the bits of $x$ $\equiv$ replace $\underline{x}$ by $\underline{2^n - 1 - x}$

    Then add 1 to get $2^n - x$

# Basic Applications of mod

- Hashing
- Pseudo random number generation
- Simple cipher

# Hashing

**Scenario:**

Map a small number of data values from a large domain $\{0, 1, \ldots, M-1\}$ …

…into a small set of locations $\{0,1, \ldots, n-1\}$ so one can quickly check if some value is present

- $\text{hash}(x) = x \bmod p$ for $p$ a prime close to $n$
  - or $\text{hash}(x) = (ax + b) \bmod p$
- Depends on all of the bits of the data
  - helps avoid collisions due to similar values
  - need to manage them if they occur

# Pseudo-Random Number Generation

## Linear Congruential method

$$x_{n+1} = (a\, x_n + c) \bmod m$$

Choose random $x_0$, $a$, $c$, $m$ and produce a long sequence of $x_n$'s

# Simple Ciphers

- **Caesar cipher**, A = 1, B = 2, . . .
  - HELLO WORLD
- **Shift cipher**
  - $f(p) = (p + k) \bmod 26$
  - $f^{-1}(p) = (p - k) \bmod 26$
- **More general**
  - $f(p) = (ap + b) \bmod 26$

# Primality

An integer *p* greater than 1 is called *prime* if the only positive factors of *p* are 1 and *p*.

A positive integer that is greater than 1 and is not prime is called *composite*.

# Fundamental Theorem of Arithmetic

Every positive integer greater than 1 has a unique prime factorization

48 = 2 • 2 • 2 • 2 • 3
591 = 3 • 197
45,523 = 45,523
321,950 = 2 • 5 • 5 • 47 • 137
1,234,567,890 = 2 • 3 • 3 • 5 • 3,607 • 3,803

# Euclid’s Theorem

**There are an infinite number of primes.**

Proof by contradiction:

Suppose that there are only a finite number of primes and call the full list $p_1, p_2, \ldots, p_n$.

$P = p_1 \cdot p_2 \cdot p_3 \cdot \ldots \cdot p_n$ ~~+~~ $p_1$

$Q = P + 1$

can't be divisible by any of $p_1 \ldots p_n$

Next multiple of $p_1$ is $P + p_1 > Q$

$p_2$ is $P + p_2 > Q$

$\vdots$

$P + p_n > Q$

# Euclid’s Theorem

**There are an infinite number of primes.**

Proof by contradiction:

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Define the number $P = p_1 \cdot p_2 \cdot p_3 \cdot \cdots \cdot p_n$ and let

$$Q = P + 1.$$

# Euclid’s Theorem

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Proof by contradiction:

Suppose that there are only a finite number of primes and call the full list $p_1, p_2, \ldots, p_n$.

Define the number $P = p_1 \cdot p_2 \cdot p_3 \cdot \cdots \cdot p_n$ and let

$$Q = P + 1.$$

Case 1: $Q$ is prime: Then $Q$ is a prime different from all of $p_1, p_2, \ldots, p_n$ since it is bigger than all of them.

Case 2: $Q > 1$ is not prime: Then $Q$ has some prime factor $p$ (which must be in the list). Therefore $p|P$ and $p|Q$ so $p|(Q - P)$ which means that $p|1$.

Both cases are contradictions so the assumption is false. ∎

# Famous Algorithmic Problems

- Primality Testing
  - Given an integer $n$, determine if $n$ is prime ✓
- Factoring
  - Given an integer $n$, determine the prime factorization of $n$

# Factoring

**Factor the following 232 digit number** [RSA768]:

123018668453011775513049495838496272077
285356959533479219732245215172640050726
365751874520219978646938995647494277406
384592519255732630345373154826850791702
612214291346167042921431160222124047927
4737794080665351419597459856902143413

1230186684530117755130494958384962720772853569595334792197322452151726400507263657518745202199786469389956474942774063845925192557326303453731548268507917026122142913461670429214311602221240479274737794080665351419597459856902143413

=

33478071698956898786044169848212690817704794983713768568912431388982883793878002287614711652531743087737814467999489

×

36746043666799590428244633799627952632279158164343087642676032283815739666511279233373417143396810270092798736308917

# Greatest Common Divisor

GCD(a, b):

**Largest integer $d$ such that $d \mid a$ and $d \mid b$**

- GCD(100, 125) = 25
- GCD(17, 49) = 1
- GCD(11, 66) = 11
- GCD(13, 0) = 13
- GCD(180, 252) = 36

3·3·4·5

4·7

3·3·[illegible]

$$\begin{array}{r} 28 \\ 9\overline{)252} \end{array}$$

# GCD and Factoring

$a = 2^3 \bullet 3 \bullet 5^2 \bullet 7 \bullet 11 = 46{,}200$

$b = 2 \bullet 3^2 \bullet 5^3 \bullet 7 \bullet 13 = 204{,}750$

$GCD(a, b) = 2^{\min(3,1)} \bullet 3^{\min(1,2)} \bullet 5^{\min(2,3)} \bullet 7^{\min(1,1)} \bullet 11^{\min(1,0)} \bullet 13^{\min(0,1)}$

**Factoring is expensive!**

**Can we compute GCD(a,b) without factoring?**

# Useful GCD Fact

If *a* and *b* are positive integers, then

$$\gcd(a,b) = \gcd(b, \underline{a \bmod b})$$

d

Proof idea: Defn of mod.

$$\underline{a} = q \cdot \underline{b} + (a \bmod b)$$

e e

If $d \mid a$ and $d \mid b$

$a \bmod b = a - q \cdot b$

$d \mid (a \bmod b)$ $\Rightarrow$ ~~d~~ a mod b

Since $d \mid a$ and $d \mid b$

$\underline{d \mid (a - q b)}$

If $e \mid b$ and $e \mid (a \bmod b)$

# Useful GCD Fact

If *a* and *b* are positive integers, then

gcd(*a*,*b*) = gcd(*b*, *a* mod *b*)

**Proof:**

By definition of mod, $a = qb + (a \bmod b)$ for some integer $q = a \text{ div } b$.

Let $d = \gcd(a, b)$. Then $d|a$ and $d|b$ so $a = kd$ and $b = jd$ for some integers $k$ and $j$.

Therefore $(a \bmod b) = a - qb = kd - qjd = (k - qj)d$.
So, $d|(a \bmod b)$ and since $d|b$ we must have $d \leq \gcd(b, a \bmod b)$.

Now, let $e = \gcd(b, a \bmod b)$. Then $e|b$ and $e\,|(a \bmod b)$ so $b = me$ and $(a \bmod b) = ne$ for some integers $m$ and $n$.

Therefore $a = qb + (a \bmod b) = qme + ne = (qm + n)e$.
So, $e|a$ and since $e|b$ we must have $e \leq \gcd(a, b)$.

It follows that $\gcd(a, b) = \gcd(b, a \bmod b)$. ■

# Another simple GCD fact

If a is a positive integer, gcd($a$,$0$) = $a$.

# Euclid’s Algorithm

gcd(a, b) = gcd(b, a mod b)

```
int gcd(int a, int b){ /* a >= b, b > 0 */
    if (b == 0) {
        return a;
    }
    else {
        return gcd(b, a % b);
    }
```

Example: GCD(660, 126)

# Euclid's Algorithm

**Repeatedly use $\gcd(a,b) = \gcd(\underline{b}, a \bmod b)$ to reduce numbers until you get $\gcd(g,0) = g$.**

gcd(660,126) = $\gcd(126, 666 \bmod 126)$

$= \gcd(126, 30)$

$= \gcd(30, 126 \bmod 30)$

$= \gcd(30, 6)$

$= \gcd(6, 0)$

$= 6$

$660 = 5 \cdot 126 + 30$

$126 = 4 \cdot 30 + 6$

# Euclid's Algorithm

**Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.**

gcd(660,126) = gcd(126, 660 mod 126) = gcd(126, 30)
= gcd(30, 126 mod 30) = gcd(30, 6)
= gcd(6, 30 mod 6) = gcd(6, 0)
= 6

# Bézout’s theorem

If $a$ and $b$ are positive integers, then there exist integers $\boldsymbol{s}$ and $\boldsymbol{t}$ such that

$$\gcd(a,b) = \boldsymbol{s}a + \boldsymbol{t}b.$$

# Extended Euclidean algorithm

- Can use Euclid's Algorithm to find $s, t$ such that

$$\gcd(a, b) = sa + tb$$

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- Can use Euclid's Algorithm to find $s, t$ such that

$$\gcd(a, b) = sa + tb$$

**Step 1 (Compute GCD & Keep Intermediary Information):**

$$\gcd(\overset{a}{35}, \overset{b}{27}) = \gcd(\overset{b}{27}, \overset{a \bmod b = r}{35 \bmod 27}) = \gcd(\overset{b}{27}, \overset{r}{8}) \quad (\overset{a\ =\ q\ *\ b\ +\ r}{35 = 1 * 27 + 8})$$

# Extended Euclidean algorithm

- **Can use Euclid's Algorithm to find $s, t$ such that**

$$\gcd(a, b) = sa + tb$$

**Step 1 (Compute GCD & Keep Intermediary Information):**

a b b a mod b = r b r

$$\begin{aligned} \gcd(35, 27) &= \gcd(27, 35 \bmod 27) = \gcd(27, 8) \\ &= \gcd(8, 27 \bmod 8) = \gcd(8, 3) \\ &= \gcd(3, 8 \bmod 3) = \gcd(3, 2) \\ &= \gcd(2, 3 \bmod 2) = \gcd(2, 1) \\ &= \gcd(1, 2 \bmod 1) = \underline{\gcd(1, 0)} \end{aligned}$$

a = q * b + r

$$\begin{aligned} &(35 = 1 * 27 + 8) \\ &(27 = 3 * 8 + 3) \\ &(8 = 2 * 3 + 2) \\ &(3 = 1 * 2 + 1) \end{aligned}$$

gcd

# Extended Euclidean algorithm

- Can use Euclid's Algorithm to find $s, t$ such that

$$\gcd(a, b) = sa + tb$$

**Step 2 (Solve the equations for r):**

a = q * b + r

$$35 = 1 * 27 + 8$$
$$27 = 3 * 8 \quad + 3$$
$$8 \quad = 2 * 3 \quad + 2$$
$$3 \quad = 1 * 2 \quad + 1$$

r = a – q * b

$$8 = 35 – 1 * 27$$

# Extended Euclidean algorithm

- Can use Euclid’s Algorithm to find $s, t$ such that

$$\gcd(a, b) = sa + tb$$

**Step 2 (Solve the equations for r):**

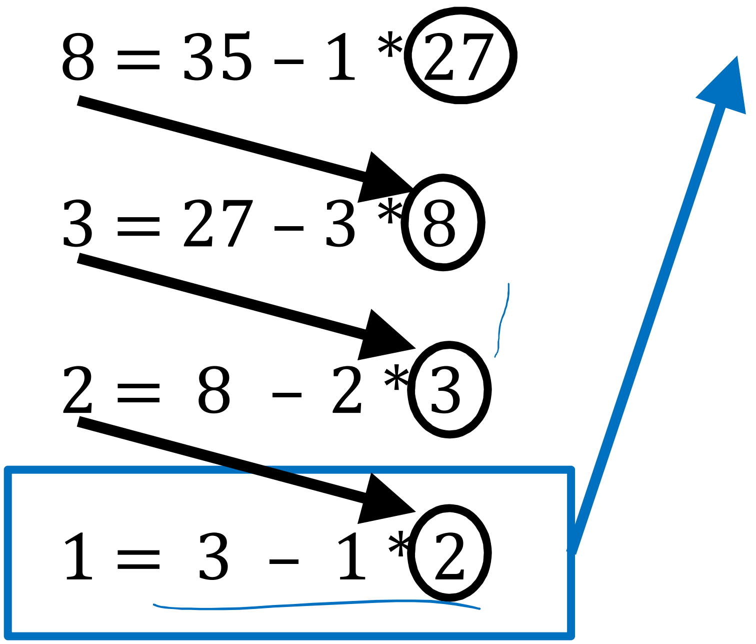
| a = q * b + r | r = a – q * b |
|---|---|
| $35 = 1 * 27 + 8$ | $8 = 35 - 1 * 27$ |
| $27 = 3 * 8 + 3$ | $3 = 27 - 3 * 8$ |
| $8 = 2 * 3 + 2$ | $2 = 8 - 2 * 3$ |
| $3 = 1 * 2 + 1$ | $1 = 3 - 1 * 2$ |
| $2 = 2 * 1 + 0$ | |

# Extended Euclidean algorithm

- Can use Euclid's Algorithm to find $s, t$ such that

$$\gcd(a, b) = sa + tb$$

**Step 3 (Backward Substitute Equations):**



$8 = 35 - 1 * \textcircled{27}$

$3 = 27 - 3 * \textcircled{8}$

$2 = 8 - 2 * \textcircled{3}$

$\boxed{1 = \underline{3 - 1 * \textcircled{2}}}$

Plug in the def of 2

$$\begin{aligned} 1 &= 3 - 1 * (8 - 2 * 3) \\ &= 3 - 8 + 2 * 3 \\ &= (-1) * 8 + 3 * 3 \end{aligned}$$

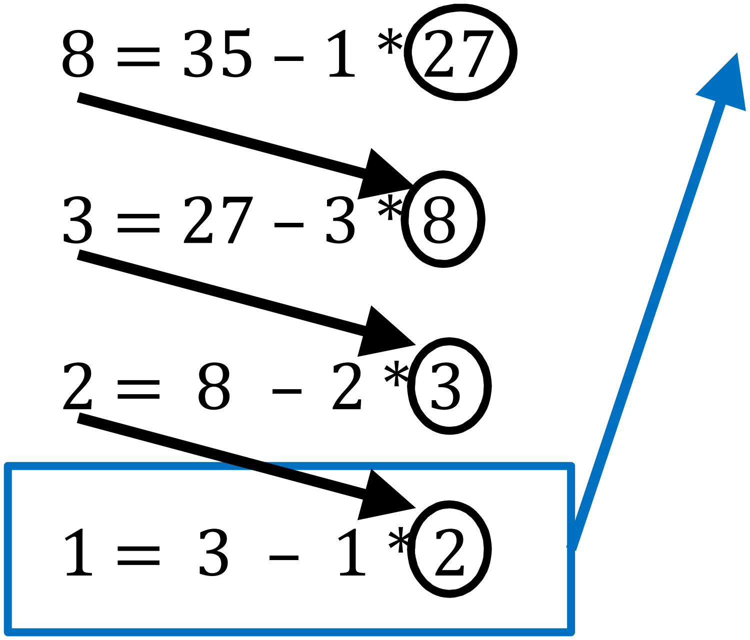
Re-arrange into 3's and 8's

# Extended Euclidean algorithm

- Can use Euclid’s Algorithm to find $s, t$ such that

$$\gcd(a, b) = sa + tb$$

**Step 3 (Backward Substitute Equations):**



$8 = 35 - 1 * \boxed{27}$

$3 = 27 - 3 * \boxed{8}$

$2 = 8 - 2 * \boxed{3}$

$1 = 3 - 1 * \boxed{2}$

Plug in the def of 2

$$1 = 3 - 1 * (8 - 2 * 3)$$
$$= 3 - 8 + 2 * 3$$
$$= (-1) * 8 + 3 * 3$$

Re-arrange into 3’s and 8’s

Plug in the def of 3

$$= (-1) * 8 + 3 * (27 - 3 * 8)$$
$$= (-1) * 8 + 3 * 27 + (-9) * 8$$
$$= 3 * 27 + (-10) * 8$$

Re-arrange into 8’s and 27’s

$$= 3 * 27 + (-10) * (35 - 1 * 27)$$
$$= 3 * 27 + (-10) * 35 + 10 * 27$$
$$= \underline{13} * 27 + \underline{(-10)} * 35$$

Re-arrange into 27’s and 35’s

# multiplicative inverse $\bmod m$

Suppose $\mathrm{GCD}(a, m) = 1$

By Bézout's Theorem, there exist integers $s$ and $t$ such that $sa + tm = 1$.

$s \bmod m$ is the multiplicative inverse of $a$:

$$1 = (sa + tm) \bmod m = sa \bmod m$$

# Example

**Solve:** $7x \equiv 1 \pmod{26}$

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**Solve:** $7x \equiv 1 \pmod{26}$

$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = 1$$

$$\begin{array}{rcl@{\qquad\qquad}rcl}
26 & = & 7 \ast 3 + 5 & 5 & = & 26 - 7 \ast 3 \\
7 & = & 5 \ast 1 + 2 & 2 & = & 7 - 5 \ast 1 \\
5 & = & 2 \ast 2 + 1 & 1 & = & 5 - 2 \ast 2
\end{array}$$

$$\begin{aligned}
1 &= 5 - 2 \ast (7 - 5 \ast 1) \\
&= (-7) \ast 2 + 3 \ast 5 \\
&= (-7) \ast 2 + 3 \ast (26 - 7 \ast 3) \\
&= (-11) \ast 7 + 3 \ast 26
\end{aligned}$$

Multiplicative inverse of 7 mod 26

**Now** $(-11) \bmod 26 = 15$. **So,** $x = 15 + 26k$ **for** $k \in \mathbb{Z}$.

# Example of a more general equation

Now solve: $7y \equiv 3 \pmod{26}$

We already computed that $15$ is the multiplicative inverse of $7$ modulo $26$:

That is, $7 \cdot 15 \equiv 1 \pmod{26}$

By the multiplicative property of mod we have

$$7 \cdot 15 \cdot 3 \equiv 3 \pmod{26}$$

So any $y \equiv 15 \cdot 3 \pmod{26}$ is a solution.

That is, $y = 19 + 26k$ for any integer $k$ is a solution.

# Math mod a prime is especially nice

$\gcd(a, m) = 1$ if $m$ is prime and $0 < a < m$ so can always solve these equations mod a prime.

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 0 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 0 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 0 | 6 | 5 | 4 | 3 | 2 | 1 |

**mod 7**