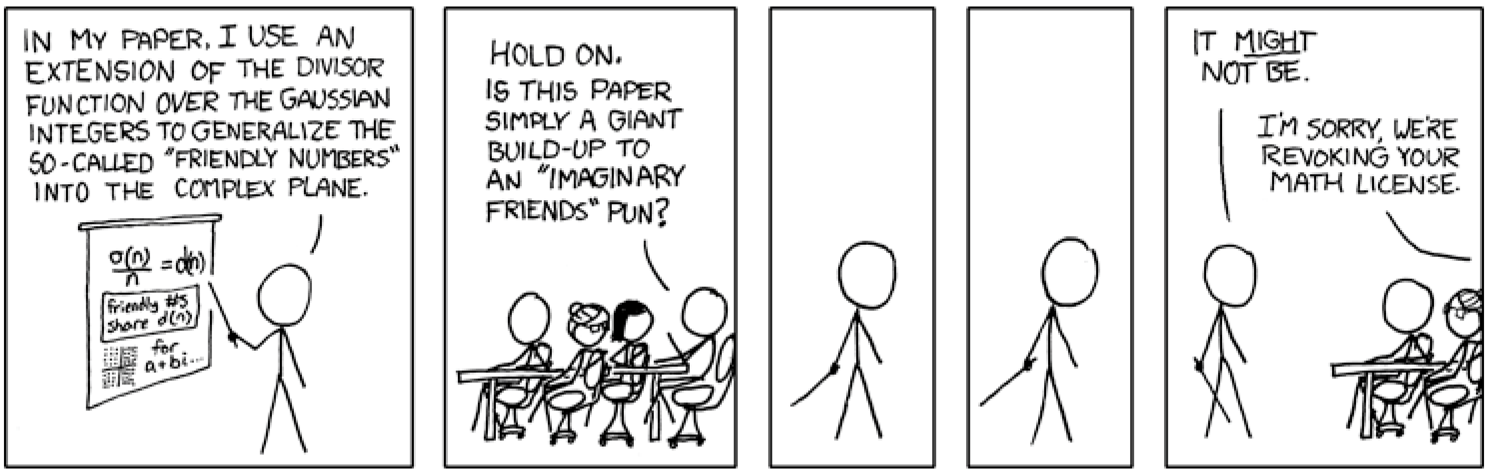
CSE 311: Foundations of Computing

Lecture 11: Modular Arithmetic and Applications



$\cap \longleftrightarrow \wedge$

$\cup \longleftrightarrow \vee$

Divisibility

Definition: "a divides b"

For $a \in \mathbb{Z}, b \in \mathbb{Z}$ with $a \neq 0$:

$$a \mid b \leftrightarrow \exists k \in \mathbb{Z} \ (b = ka)$$

Check Your Understanding. Which of the following are true?

5 | 1	25 | 5	5 | 0	3 | 2
5 | 1 iff 1 = 5k	25 | 5 iff 5 = 25k	5 | 0 iff 0 = 5k	3 | 2 iff 2 = 3k
1 | 5	5 | 25	0 | 5	2 | 3
1 | 5 iff 5 = 1k	5 | 25 iff 25 = 5k	0 | 5 iff 5 = 0k	2 | 3 iff 3 = 2k

Division Theorem

Division Theorem

For $a \in \mathbb{Z}, d \in \mathbb{Z}$ with $d > 0$
there exist *unique* integers *q*, *r* with $0 \leq r < d$
such that $a = dq + r$.

quotient – remainder

To put it another way, if we divide *d* into *a*, we get a unique quotient $q = a$ **div** d
and non-negative remainder $r = a$ **mod** d

$25 = 3 \cdot 7 + 4$ $\quad$ $25 \bmod 7 = 4$

$-2 = (-1) \cdot 7 + 5$ $\quad$ $(-2) \bmod 7 = 5$

Note: r ≥ 0 even if a < 0.
Not quite the same as `a%d`.

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```
public class Test2 {
    public static void main(String args[]) {
        int a = -5;
        int d = 2;
        System.out.println(a % d);
    }
}
```

```
 ----jGRASP exec: java Test2
-1

 ----jGRASP: operation complete.
```

Note: $r \geq 0$ even if $a < 0$.
Not quite the same as **`a%d`**.

Arithmetic, mod 7

$$a +_7 b = (a + b) \bmod 7$$

$$a \times_7 b = (a \times b) \bmod 7$$

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

$5 \cdot 4 = 20$

$20 \bmod 7 = 6$

Modular Arithmetic

Definition: "a is congruent to b modulo m"

For $a, b, m \in \mathbb{Z}$ with $m > 0$

$$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$$

Check Your Understanding. What do each of these mean? When are they true?

x ≡ 0 (mod 2)

x is even

-1 ≡ 19 (mod 5)

true

y ≡ 2 (mod 7)

#'s of the form 2+7k

-5, 2, 9, 16 ...

Modular Arithmetic

Definition: "a is congruent to b modulo m"

For $a, b, m \in \mathbb{Z}$ with $m > 0$

$$a \equiv b \pmod{m} \leftrightarrow m \mid (a - b)$$

Check Your Understanding. What do each of these mean? When are they true?

x ≡ 0 (mod 2)

This statement is the same as saying "x is even"; so, any x that is even (including negative even numbers) will work.

-1 ≡ 19 (mod 5)

This statement is true. 19 - (-1) = 20 which is divisible by 5

y ≡ 2 (mod 7)

This statement is true for y in { ..., -12, -5, 2, 9, 16, ...}. In other words, all y of the form 2+7k for k an integer.

Modular Arithmetic: A Property

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ be integers with $\boldsymbol{m} > \mathbf{0}$.
Then, $\boldsymbol{a} \equiv \boldsymbol{b} \ (\mathbf{mod}\ \boldsymbol{m})$ if and only if $\boldsymbol{a}\ \mathbf{mod}\ \boldsymbol{m} = \boldsymbol{b}\ \mathbf{mod}\ \boldsymbol{m}$.

Suppose that $a \equiv b \pmod{m}$.

$\therefore m \mid (a-b)$ $\quad \therefore a-b = km$ for some integer k

$\therefore a = b+km$ $\quad$ By Division Thm $b = qm+r$ $\quad 0 \le r < m$

(r = b mod m)

$\therefore a = (k+q)m + r$

$\therefore a \bmod m = r = b \bmod m$

Suppose that $a \bmod m = b \bmod m$.

$\therefore$ Let $r = a \bmod m = b \bmod m$ $\quad a = qm+r \quad 0 \le r < m$ for some integer q

$b = sm+r$ for some integer s.

$\therefore a-b = (q-s)m$

$\therefore m \mid (a-b)$ $\quad \therefore a \equiv b \pmod{m}$

Modular Arithmetic: A Property

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ be integers with $\boldsymbol{m > 0}$.
Then, $\boldsymbol{a \equiv b \ (\mathbf{mod}\ m)}$ if and only if $\boldsymbol{a \ \mathbf{mod}\ m = b \ \mathbf{mod}\ m}$.

Suppose that $a \equiv b \pmod{m}$.

Then, $m \mid (a - b)$ by definition of congruence.

So, $a - b = km$ for some integer k by definition of divides.

Therefore, $a = b + km$.

Taking both sides modulo m we get:

$$a \bmod m = (b + km) \bmod m = b \bmod m.$$

Suppose that $a \bmod m = b \bmod m$.

By the division theorem, $a = mq + (a \bmod m)$ and

$b = ms + (b \bmod m)$ for some integers q,s.

Then, $a - b = (mq + (a \bmod m)) - (ms + (b \bmod m))$

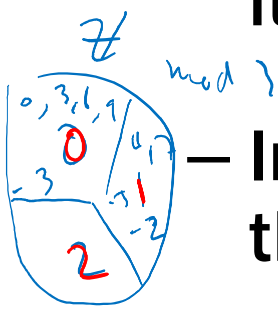
$= m(q - s) + (a \bmod m - b \bmod m)$

$= m(q - s)$ since $a \bmod m = b \bmod m$

Therefore, $m \mid (a - b)$ and so $a \equiv b \pmod{m}$.

The $\operatorname{mod} m$ function vs the $\equiv \pmod{m}$ predicate

- What we have just shown
  - The $\operatorname{mod} m$ function takes any $a \in \mathbb{Z}$ and maps it to a remainder $a \operatorname{mod} m \in \{0,1,..,m-1\}$.



  - Imagine grouping together all integers that have the same value of the $\operatorname{mod} m$ function

    That is, the same remainder in $\{0,1,..,m-1\}$.

  - The $\equiv \pmod{m}$ predicate compares $a, b \in \mathbb{Z}$. It is true if and only if the $\operatorname{mod} m$ function has the same value on a and on b.

    That is, a and b are in the same group.

Modular Arithmetic: Addition Property

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$ —

Proof: Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$

$\therefore$ $m \mid (a-b)$ and $m \mid (c-d)$

$\therefore$ $a-b = km$ for some integer k.

$\therefore$ $c-d = sm$ for some integer s.

$\therefore$ $a-b+c-d = km+sm$

$(a+c)-(b+d) = km + sm$

$\therefore$ $= (k+s)m$

$\therefore$ $m \mid ((a+c)-(b+d))$

$\therefore$ $a+c \equiv b+d \pmod{m}$

∎

Modular Arithmetic: Addition Property

Let $\boldsymbol{m}$ be a positive integer. If $\boldsymbol{a \equiv b \pmod{m}}$ and $\boldsymbol{c \equiv d \pmod{m}}$, then $\boldsymbol{a + c \equiv b + d \pmod{m}}$

Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. Unrolling definitions gives us some k such that $a - b = km$, and some j such that $c - d = jm$.

Adding the equations together gives us $(a + c) - (b + d) = m(k + j)$. Now, re-applying the definition of congruence gives us $a + c \equiv b + d \pmod{m}$.

Modular Arithmetic: Multiplication Property

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$

Proof: Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$

$\therefore$ $a - b = km$ for some integer k

$c - d = sm$ for some integer s.

$\therefore$ $a = b + km$

and $c = d + sm$

$\therefore$ $a \cdot c = (b+km)(d+sm) = bd + bsm + kmd + ksm^2$

$= bd + m(bs + kd + ksm)$ ← integer

$\therefore$ $m \mid (ac - bd)$

$\therefore$ $ac \equiv bd \pmod{m}$ ∎

Modular Arithmetic: Multiplication Property

Let $\boldsymbol{m}$ be a positive integer. If $\boldsymbol{a} \equiv \boldsymbol{b} \pmod{\boldsymbol{m}}$ and $\boldsymbol{c} \equiv \boldsymbol{d} \pmod{\boldsymbol{m}}$, then $\boldsymbol{ac} \equiv \boldsymbol{bd} \pmod{\boldsymbol{m}}$

Suppose that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. Unrolling definitions gives us some k such that $a - b = km$, and some j such that $c - d = jm$.

Then, $a = km + b$ and $c = jm + d$. Multiplying both together gives us $ac = (km + b)(jm + d) = kjm^2 + kmd + bjm + bd$.

Re-arranging gives us $ac - bd = m(kjm + kd + bj)$.
Using the definition of congruence gives us $ac \equiv bd \pmod{m}$.

Example

Let $\boldsymbol{n}$ be an integer.
Prove that $\boldsymbol{n^2 \equiv 0 \ (\mathrm{mod}\ 4)}$ or $\boldsymbol{n^2 \equiv 1 \ (\mathrm{mod}\ 4)}$

Let's start by looking a a small example:

$0^2 = 0 \equiv 0 \pmod 4$
$1^2 = 1 \equiv 1 \pmod 4$
$2^2 = 4 \equiv 0 \pmod 4$
$3^2 = 9 \equiv 1 \pmod 4$
$4^2 = 16 \equiv 0 \pmod 4$

Example

Let $\boldsymbol{n}$ be an integer.
Prove that $\boldsymbol{n^2 \equiv 0 \pmod{4}}$ or $\boldsymbol{n^2 \equiv 1 \pmod{4}}$

Case 1 (n is even):

$\therefore n \equiv 0 \pmod{2}$

$\therefore n = 2k$ for some integer k

$\therefore n^2 = 4k^2$

$\therefore n^2 \equiv 0 \pmod{4}$

Case 2 (n is odd):

$\therefore n \equiv 1 \pmod{2}$

$\therefore n = 2k+1$ for some integer k

$\therefore n^2 = (2k+1)^2 = 4k^2 + 4k + 1$
$= 4(k^2+k) + 1$

$\therefore n^2 \equiv 1 \pmod{4}$

Let's start by looking a a small example:

$0^2 = 0 \equiv 0 \pmod{4}$
$1^2 = 1 \equiv 1 \pmod{4}$
$2^2 = 4 \equiv 0 \pmod{4}$
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$4^2 = 16 \equiv 0 \pmod{4}$

It looks like

$n \equiv 0 \pmod{2} \rightarrow n^2 \equiv 0 \pmod{4}$, and
$n \equiv 1 \pmod{2} \rightarrow n^2 \equiv 1 \pmod{4}$.

Example

Let $\boldsymbol{n}$ be an integer.
Prove that $\boldsymbol{n^2 \equiv 0 \pmod 4}$ or $\boldsymbol{n^2 \equiv 1 \pmod 4}$

Let's start by looking a a small example:

$0^2 = 0 \equiv 0 \pmod 4$
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It looks like
$n \equiv 0 \pmod 2 \rightarrow n^2 \equiv 0 \pmod 4$, and
$n \equiv 1 \pmod 2 \rightarrow n^2 \equiv 1 \pmod 4$.

Case 1 (n is even):

Suppose $n \equiv 0 \pmod 2$.
Then, $n = 2k$ for some integer k.
So, $n^2 = (2k)^2 = 4k^2$. So, by definition of congruence,
$n^2 \equiv 0 \pmod 4$.

Case 2 (n is odd):

Suppose $n \equiv 1 \pmod 2$.
Then, $n = 2k + 1$ for some integer k.
So, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$.
So, by definition of congruence, $n^2 \equiv 1 \pmod 4$.

n-bit Unsigned Integer Representation

- Represent integer x as sum of powers of 2:

  If $\sum_{i=0}^{n-1} b_i 2^i$ where each $b_i \in \{0,1\}$

  then representation is $b_{n-1}...b_2\, b_1\, b_0$

  99 = 64 + 32 + 2 + 1

  18 = 16 + 2

- For n = 8:

  99: 0110 0011

  18: 0001 0010

Sign-Magnitude Integer Representation

n-bit signed integers

Suppose that $-2^{n-1} < x < 2^{n-1}$

First bit as the sign, $n - 1$ bits for the value

99 = 64 + 32 + 2 + 1

18 = 16 + 2

For n = 8:

99: 0110 0011

-18: 1001 0010

Any problems with this representation?

Two's Complement Representation

n bit signed integers, first bit will still be the sign bit

Suppose that $0 \leq x < 2^{n-1}$,
x is represented by the binary representation of x
Suppose that $0 \leq x \leq 2^{n-1}$,
$-x$ is represented by the binary representation of $2^n - x$

Key property: Twos complement representation of any number $\boldsymbol{y}$ is equivalent to $\boldsymbol{y} \ \mathbf{mod}\ \mathbf{2^n}$ so arithmetic works $\mathbf{mod}\ \mathbf{2^n}$

99 = 64 + 32 + 2 + 1
18 = 16 + 2

For n = 8:
99: 0110 0011
-18: 1110 1110

Sign-Magnitude vs. Two's Complement

-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1111	1110	1101	1100	1011	1010	1001	0000	0001	0010	0011	0100	0101	0110	0111

Sign-bit

-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
1000	1001	1010	1011	1100	1101	1110	1111	0000	0001	0010	0011	0100	0101	0110	0111

Two's complement

Two's Complement Representation

- For $0 < x \leq 2^{n-1}$, $-x$ is represented by the binary representation of $2^n - x$
  - That is, the two's complement representation of any number y has the same value as y modulo 2^n.

- To compute this: Flip the bits of x then add 1:
  - All 1's string is $2^n - 1$, so

    Flip the bits of x $\equiv$ replace x by $2^n - 1 - x$

    Then add 1 to get $2^n - x$

Basic Applications of mod

- Hashing
- Pseudo random number generation
- Simple cipher

Hashing

Scenario:

Map a small number of data values from a large domain $\{0, 1, \ldots, M-1\}$ …

…into a small set of locations $\{0,1, \ldots, n-1\}$ so one can quickly check if some value is present

- $\text{hash}(x) = x \bmod p$ **for** p **a prime close to** n
  - **or** $\text{hash}(x) = (ax + b) \bmod p$
- **Depends on all of the bits of the data**
  - **helps avoid collisions due to similar values**
  - **need to manage them if they occur**

Pseudo-Random Number Generation

Linear Congruential method

$$x_{n+1} = (a\, x_n + c) \bmod m$$

Choose random x_0, a, c, m and produce a long sequence of x_n's

Simple Ciphers

- **Caesar cipher**, A = 1, B = 2, . . .
  - HELLO WORLD
- **Shift cipher**
  - $f(p) = (p + k) \bmod 26$
  - $f^{-1}(p) = (p - k) \bmod 26$
- **More general**
  - $f(p) = (ap + b) \bmod 26$