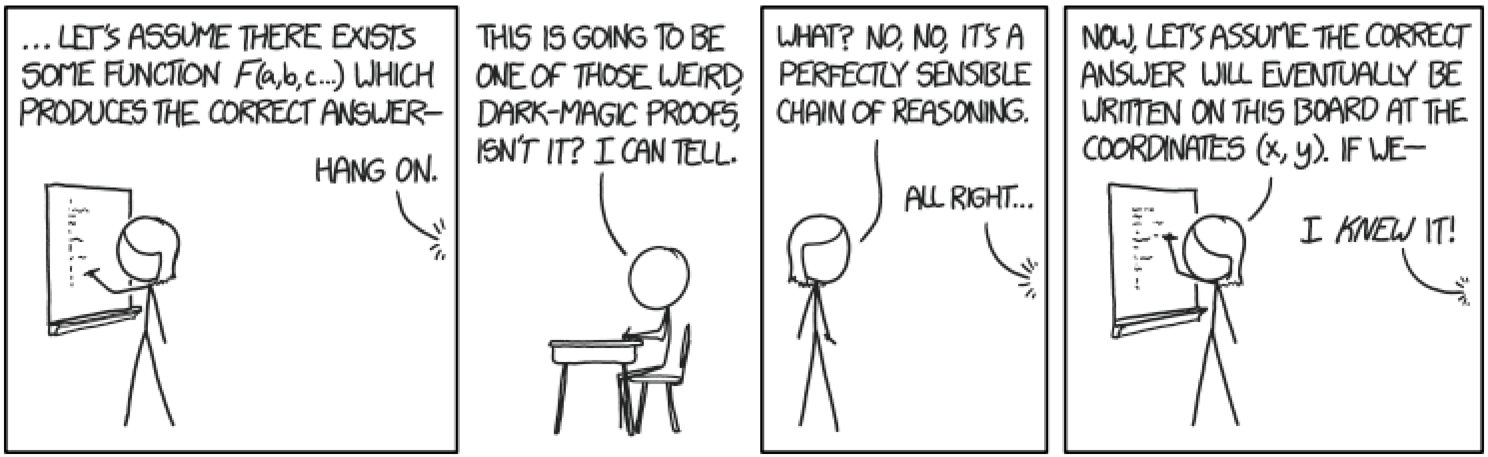
CSE 311: Foundations of Computing

Lecture 9: English Proofs, Strategies, Set Theory



Last class: Inference Rules for Quantifiers

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore \; P(a) \textbf{ for any } a}$$

Intro ∀

$$\frac{\text{“}\textbf{Let } a \textbf{ be arbitrary*}\text{”}\ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

* in the domain of P. No other name in P depends on a

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore \; P(c) \textbf{ for some } \textbf{\textit{special}}\text{**}\; c}$$

** c is a NEW name.
List all dependencies for c.

Last class: Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove: "The square of every even number is even."

Formal proof of: $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2.1	Even(**a**)	Assumption
2.2	$\exists y\ (\mathbf{a} = 2y)$	Definition of Even
2.3	$\mathbf{a} = 2\mathbf{b}$	Elim $\exists$: **b** special depends on **a**
2.4	$\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$	Algebra
2.5	$\exists y\ (\mathbf{a}^2 = 2y)$	Intro $\exists$ rule
2.6	Even($\mathbf{a}^2$)	Definition of Even

2. $\text{Even}(\mathbf{a}) \rightarrow \text{Even}(\mathbf{a}^2)$ — Direct proof rule
3. $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro $\forall$: 1,2

English Proof: Even and Odd

$\text{Even}(x) \equiv \exists y\ (x=2y)$
$\text{Odd}(x) \equiv \exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Proof: Let a be an arbitrary even integer.	1. **Let a be an arbitrary integer** 2.1 Even(a) Assumption
Then, by definition, $a = 2b$ for some integer b (depending on a).	2.2 $\exists y\ (a = 2y)$ Definition 2.3 $a = 2b$ **b** special depends on **a**
Squaring both sides, we get $a^2 = 4b^2 = 2(2b^2)$.	2.4 $a^2 = 4b^2 = 2(2b^2)$ Algebra
Since $2b^2$ is an integer, by definition, a^2 is even.	2.5 $\exists y\ (a^2 = 2y)$ 2.6 $\text{Even}(a^2)$ Definition
Since a was arbitrary, it follows that the square of every even number is even. ■	2. $\text{Even}(a) \rightarrow \text{Even}(a^2)$ 3. $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The square of every odd integer is odd."

Pf. Let a be arbitrary odd integer.

There exists b depending on a s.t. $2b+1 = a$.

Therefore, by algebra, $(2b+1)^2 = a^2$.

$a^2 = 4b^2 + 4b + 1 = 2(2b^2 + 2b) + 1$

Since $2b^2 + 2b$ is an integer, a^2 is odd.

a^2 is odd

Because a was arbitrary, square of every odd is odd

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The square of every odd integer is odd."

Proof: Let b be an arbitrary odd integer.

Then, b = 2c+1 for some integer c (depending on b).

Therefore, $b^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1$.

Since $2c^2+2c$ is an integer, b^2 is odd. Since b was arbitrary, the square of every odd integer is odd. ■

Proof Strategies: Counterexamples

To *disprove* $\forall x\, P(x)$ prove $\exists_x \neg P(x)$:

- Works by de Morgan's Law: $\neg\forall x\, P(x) \equiv \exists x \neg P(x)$
- All we need to do that is find a c for which $P(c)$ is **false**
- This example is called a ***counterexample*** to $\forall x\, P(x)$.

e.g. Disprove "Every prime number is odd"

Find a prime numb that is not odd

prime(2) $\wedge$ $\neg$odd(2)

$\neg \forall x,\ \text{Prime}(x) \rightarrow \text{odd}(x)$

$\equiv \exists x \neg (\text{Prime}(x) \rightarrow \text{odd}(x))$

$\equiv \exists x \neg (\neg \text{Prime}(x) \vee \text{odd}(x)) \equiv \exists x\ \text{Prime}(x) \wedge \neg \text{odd}(x)$

Proof Strategies: Proof by Contrapositive

If we assume $\neg q$ and derive $\neg p$, then we have proven $\neg q \to \neg p$, which is equivalent to proving $p \to q$.

1.1. $\neg q$ Assumption

...

1.3. $\neg p$

1. $\neg q \to \neg p$ Direct Proof Rule
2. $p \to q$ Contrapositive: 1

Proof by Contradiction: One way to prove $\neg p$

If we assume p and derive F (a contradiction), then we have proven $\neg$p.

$\neg p \equiv T \quad p = F$

Assm $p = T \rightarrow F$

Assumption is incorrect

$p = F$

$\neg p = T$

1.1. p Assumption

...

1.3. F

1. $p \rightarrow$ F Direct Proof rule
2. $\neg p \vee$ F Law of Implication: 1
3. $\neg p$ Identity: 2

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "No integer is both even and odd."

English proof: $\neg \exists x\,(\text{Even}(x) \wedge \text{Odd}(x))$

$\equiv \forall x\, \boxed{\neg(\text{Even}(x) \wedge \text{Odd}(x))}$

$P(a) \equiv T$
Assume $P(a) \equiv F$
Assume $\text{Even}(a) \wedge \text{odd}(a)$

pf. Let a be an arbitrary integer. We prove by contrad.
Assume a is even and odd. $(*)$
Since a is even $a=2b$ for some b depending on a.
Since a is odd $a=2c+1$ for some c dep on a.
Thus, $2b = 2c+1$. So $b = c + \frac{1}{2}$
No two integers differ by $\frac{1}{2}$. Contradiction. $(*)$ is False

a is not both even and odd.
Since a was arbitrary no integer is both even and odd.

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "No integer is both even and odd."

English proof: $\neg\exists x\ (\text{Even}(x) \land \text{Odd}(x))$

$\equiv \forall x\ \neg(\text{Even}(x) \land \text{Odd}(x))$

Proof: We work by contradiction. Let c be an arbitrary integer and suppose that it is both even and odd. Then c=2a for some integer a (depending on c) and c=2b+1 for some integer b (depending on c). Therefore 2a=2b+1 and hence a=b+½.

But two integers cannot differ by ½ so this is a contradiction. So, no integer is both even and odd. ■

Rational Numbers

Domain of Discourse
Real Numbers

- A real number x is *rational* iff there exist integers p and q with $q \neq 0$ such that $x = p/q$.

$$\text{Rational}(x) \equiv \exists p\, \exists q\ ((x = p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge q \neq 0)$$

$2/3$ $\quad$ $\frac{10}{2}$ $\quad$ $\cancel{\frac{3}{0}}$

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) $\equiv \exists p\ \exists q\ ((x = p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge (q \neq 0))$

Prove: "If x and y are rational then xy is rational."

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) $\equiv \exists p\, \exists q\, ((x = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$

Prove: "If x and y are rational then xy is rational."

Proof: Let x and y be rational numbers. Then, x = a/b for some integers a, b, where $b \neq 0$, and y = c/d for some integers c,d, where $d \neq 0$.

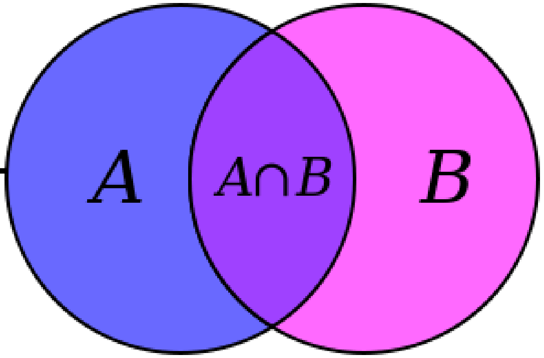
Multiplying, we get that xy = (ac)/(bd).

Since b and d are both non-zero, so is bd; furthermore, ac and bd are integers. It follows that xy is rational, by definition of rational. ∎

Proofs

- Formal proofs follow simple well-defined rules and should be easy to check
  - In the same way that code should be easy to execute
- English proofs correspond to those rules but are designed to be easier for humans to read
  - Easily checkable in principle
- Simple proof strategies already do a lot
  - Later we will cover a specific strategy that applies to loops and recursion (mathematical induction)

Set Theory



Sets are collections of objects called **elements.**

Write $\boxed{a \in B}$ to say that a is an element of set B,
and $\boxed{a \notin B}$ to say that it is not.

Some simple examples
A = {1}
B = {1, 3, 2}
C = {□, 1}
D = {{17}, 17}
E = {1, 2, 7, cat, dog, ∅, α}

Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers;** $\mathbb{N}$ = {0, 1, 2, …}
$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z}$ = {…, -2, -1, 0, 1, 2, …}
$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48
$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, π, $\sqrt{2}$
[n] is the set **{1, 2, …, n}** when **n** is a natural number
{} = ∅ is the **empty set**; the *only* set with no elements

Sets can be elements of other sets

For example

$A = \{\{1\},\{2\},\{1,2\},\varnothing\}$

$B = \{1,2\}$

Then $B \in A$.

Definitions

- A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\,(x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\,(x \in A \rightarrow x \in B)$$

- Note: $\boldsymbol{(A = B) \equiv (A \subseteq B) \wedge (B \subseteq A)}$

Definition: Equality

A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\,(x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

$C = D = E$

$F \neq E$

$3 \notin F$

$3 \in E$

Which sets are equal to each other?

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall\, x\, (x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}

QUESTIONS

$\varnothing \subseteq A$? ✓

$A \subseteq B$? ✗

$C \subseteq B$? ✓

Building Sets from Predicates

S = the set of all* x for which $P(x)$ is true

$$\boxed{S = \{x : P(x)\}}$$

S = the set of all x in A for which $P(x)$ is true

$$\boxed{S = \{x \in A : P(x)\}}$$

*in the domain of P, usually called the "universe" U

Set Operations

$$A \cup B = \{ x : (x \in A) \vee (x \in B) \}$$ **Union**

$$A \cap B = \{ x : (x \in A) \wedge (x \in B) \}$$ **Intersection**

$$A \setminus B = \{ x : (x \in A) \wedge (x \notin B) \}$$ **Set Difference**

A minus B

A = {1, 2, 3}
B = {3, 5, 6}
C = {3, 4}

QUESTIONS

Using A, B, C and set operations, make…

[6] = $A \cup B \cup C$

{3} = $A \cap C$

{1,2} = $A \setminus B$

More Set Operations

$$A \oplus B = \{ x : (x \in A) \oplus (x \in B) \}$$

Symmetric Difference

$$\bar{A} = \{ x : x \notin A \}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 2, 4, 6}
Universe:
U = {1, 2, 3, 4, 5, 6}

$A \oplus B = \{3, 4, 6\}$
$\bar{A} = \{4,5,6\}$

It’s Boolean algebra again

- Definition for $\cup$ based on $\vee$

- Definition for $\cap$ based on $\wedge$

- Complement works like $\neg$

De Morgan's Laws

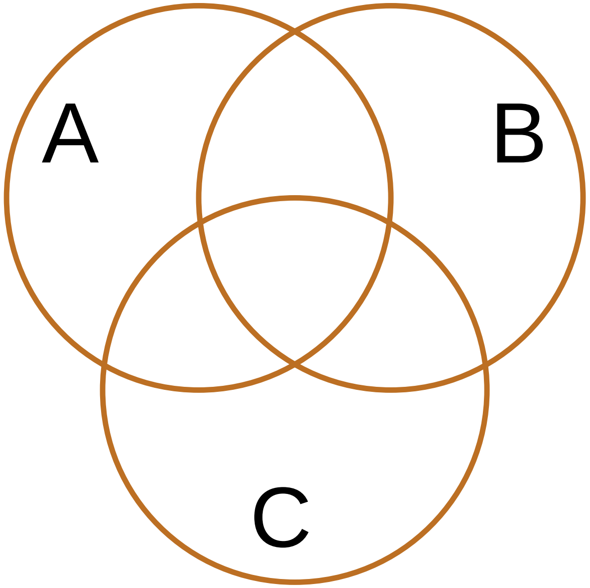
$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

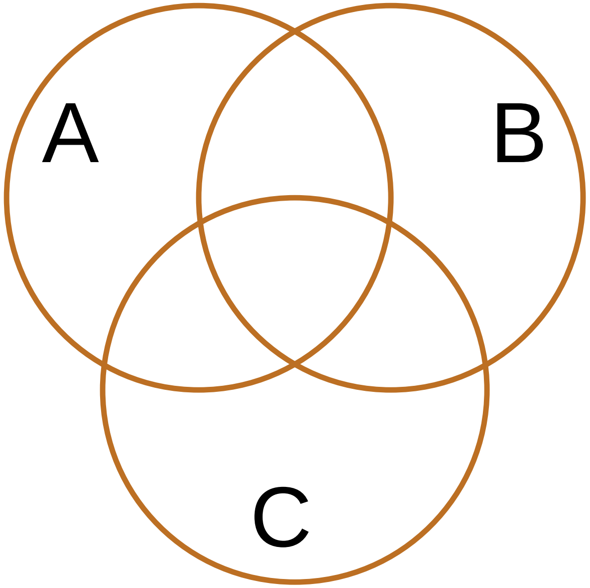
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Proof technique:
To show C = D show
$x \in$ C $\rightarrow x \in$ D and
$x \in$ D $\rightarrow x \in$ C

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$





Power Set

- **Power Set of a set A = set of all subsets of A**

$$\mathcal{P}(A) = \{\, B : B \subseteq A \,\}$$

- **e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class**

$\mathcal{P}$(Days)=?

$\mathcal{P}(\varnothing)$=?

Power Set

- **Power Set of a set A = set of all subsets of A**

$$\mathcal{P}(A) = \{\, B : B \subseteq A \,\}$$

- **e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class**

$\mathcal{P}$(Days)=$\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \varnothing\}$

$\mathcal{P}(\varnothing)=\{\varnothing\} \neq \varnothing$

Cartesian Product

$$A \times B = \{\, (a, b) : a \in A, b \in B \,\}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c),
(2,a), (2,b), (2,c)}.

$\boldsymbol{A} \times \emptyset = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \wedge \boldsymbol{b} \in \emptyset\} = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \wedge \mathbf{F}\} = \emptyset$

Representing Sets Using Bits

- Suppose universe U is $\{1,2, \ldots, n\}$
- Can represent set $B \subseteq U$ as a vector of bits:

$b_1 b_2 \ldots b_n$ where $b_i = 1$ when $i \in B$

$b_i = 0$ when $i \notin B$

  – Called the *characteristic vector* of set B

- Given characteristic vectors for A and B
  – What is characteristic vector for $A \cup B$? $A \cap B$?

UNIX/Linux File Permissions

- **`ls -l`**

```
drwxr-xr-x ... Documents/
-rw-r--r-- ... file1
```

- Permissions maintained as bit vectors
  - Letter means bit is 1
  - "–" means bit is 0.

Bitwise Operations

$$\begin{array}{rl} & 01101101 \\ \vee & \underline{00110111} \\ & 01111111 \end{array}$$

Java: `z=x|y`

$$\begin{array}{rl} & 00101010 \\ \wedge & \underline{00001111} \\ & 00001010 \end{array}$$

Java: `z=x&y`

$$\begin{array}{rl} & 01101101 \\ \oplus & \underline{00110111} \\ & 01011010 \end{array}$$

Java: `z=x^y`

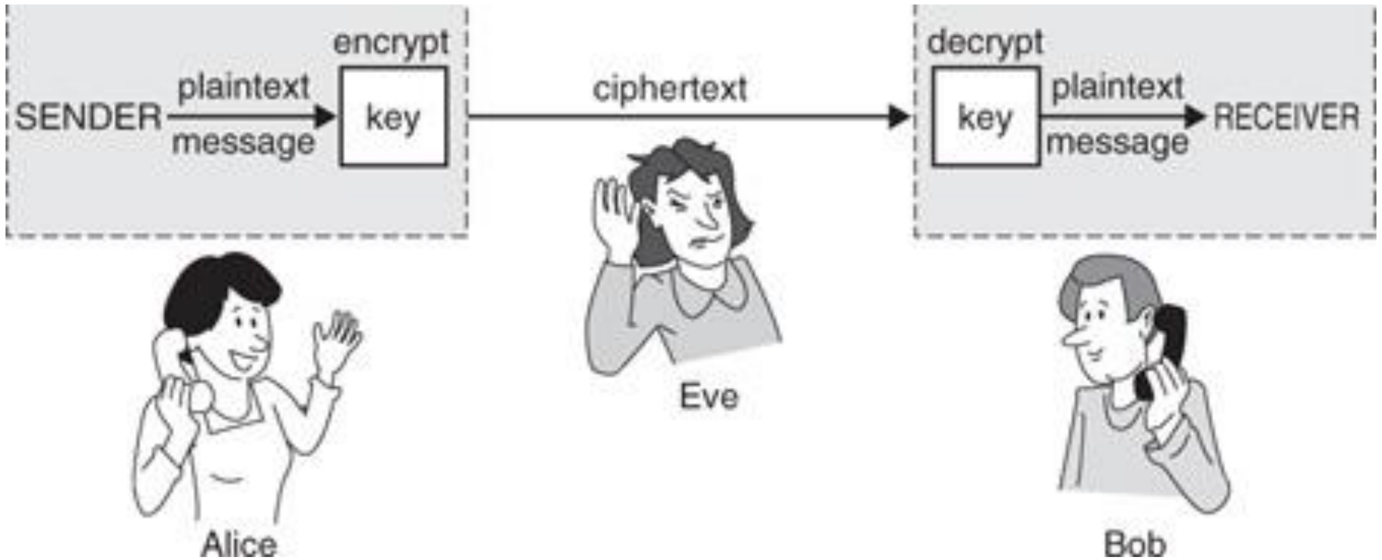
A Useful Identity

- If x and y are bits: $(x \oplus y) \oplus y = ?$

- What if x and y are bit-vectors?

Private Key Cryptography

- **Alice** wants to communicate message secretly to **Bob** so that eavesdropper **Eve** who hears their conversation cannot tell what **Alice**'s message is.
- **Alice** and **Bob** can get together and privately share a secret key K ahead of time.



One-Time Pad

- Alice and Bob privately share random n-bit vector K
  - Eve does not know K
- Later, Alice has n-bit message m to send to Bob
  - Alice computes $C = m \oplus K$
  - Alice sends C to Bob
  - Bob computes $m = C \oplus K$ which is $(m \oplus K) \oplus K$
- Eve cannot figure out m from C unless she can guess K

Russell's Paradox

$$\boxed{S = \{\, x : x \notin x \,\}}$$

Suppose for contradiction that $S \in S$...

Russell's Paradox

$$S = \{\, x : x \notin x \,\}$$

Suppose for contradiction that $S \in S$. Then, by definition of S, $S \notin S$, but that's a contradiction.

Suppose for contradiction that $S \notin S$. Then, by definition of the set S, $S \in S$, but that's a contradiction, too.

This is reminiscent of the truth value of the statement "This statement is false."