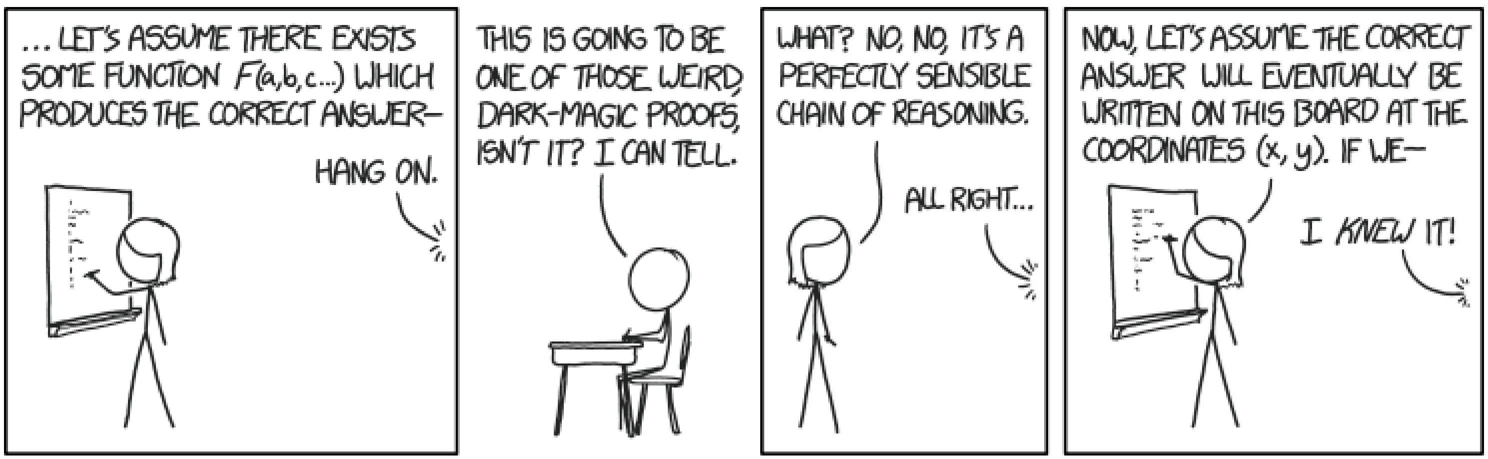
CSE 311: Foundations of Computing

Lecture 9: English Proofs, Strategies, Set Theory



Last class: Inference Rules for Quantifiers

Intro ∃

P(c) for some c

∴ ∃x P(x)

Elim ∀

∀x P(x)

∴ P(a) **for any** a

Intro ∀

“**Let** a **be arbitrary***”...P(a)

∴ ∀x P(x)

* in the domain of P. No other name in P depends on a

Elim ∃

∃x P(x)

∴ P(c) **for some *special*** ** c

** c is a NEW name.
List all dependencies for c.

all the arbitrary names in P.

Last class: Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Prove: "The square of every even number is even."

Formal proof of: $\forall x\, (\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2.1	Even(**a**)	Assumption
2.2	$\exists y\, (\mathbf{a} = 2y)$	Definition of Even
2.3	$\mathbf{a} = 2\mathbf{b}$	Elim ∃: **b** special depends on **a**
2.4	$\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$	Algebra
2.5	$\exists y\, (\mathbf{a}^2 = 2y)$	Intro ∃ rule
2.6	$\text{Even}(\mathbf{a}^2)$	Definition of Even

2. $\text{Even}(\mathbf{a}) \rightarrow \text{Even}(\mathbf{a}^2)$ — Direct proof rule
3. $\forall x\, (\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro ∀: 1,2

English Proof: Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Prove "The square of every even integer is even."

"Let a be an arbitrary integer & assume that a is even"

Proof: Let a be an arbitrary even integer.	1. Let a be an arbitrary integer	
	2.1 Even(a)	Assumption
Then, by definition, $a = 2b$ for some integer b (depending on a).	2.2 $\exists y\ (a = 2y)$	Definition
	2.3 $a = 2b$	b special depends on a
Squaring both sides, we get $a^2 = 4b^2 = 2(2b^2)$.	2.4 $a^2 = 4b^2 = 2(2b^2)$	Algebra
Since $2b^2$ is an integer, by definition, a^2 is even.	2.5 $\exists y\ (a^2 = 2y)$	
	2.6 $\text{Even}(a^2)$	Definition
Since a was arbitrary, it follows that the square of every even number is even. ■	2. $\text{Even}(a) \rightarrow \text{Even}(a^2)$	
	3. $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$	

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The square of every odd integer is odd."

Proof: Let a be an arbitrary integer that is odd.
Therefore $a = 2b+1$ for some integer b (depending on a)

$\therefore a^2 = (2b+1)^2 = 4b^2 + 4b + 1$
$= 2(2b^2+2b)+1$

$\therefore$ Since $2b^2+2b$ is an integer, a^2 is odd.

$\therefore$ Since a was an arbitrary odd integer, the square of every odd integer is odd ∎

q.e.d.

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The square of every odd integer is odd."

Proof: Let b be an arbitrary odd integer.

Then, b = 2c+1 for some integer c (depending on b).

Therefore, $b^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1$.

Since $2c^2+2c$ is an integer, b^2 is odd. Since b was arbitrary, the square of every odd integer is odd. ■

Proof Strategies: Counterexamples

To *disprove* $\forall x\, P(x)$ prove $\exists \neg P(x)$:

- Works by de Morgan's Law: $\neg \forall x\, P(x) \equiv \exists x \neg P(x)$
- All we need to do that is find an x for which $P(x)$ is false
- This example is called a *counterexample* to $\forall x\, P(x)$.

$\forall x\, (Prime(x) \to Odd(x))$

e.g. Disprove "Every prime number is odd"

Since 2 is prime but not odd

$\neg(Prime(2) \to Odd(2))$

$\therefore \neg \forall x\, (Prime(x) \to Odd(x))$

Proof Strategies: Proof by Contrapositive

If we assume $\neg q$ and derive $\neg p$, then we have proven $\neg q \rightarrow \neg p$, which is equivalent to proving $p \rightarrow q$.

1.1.	$\neg q$	Assumption
...		
1.3.	$\neg p$	
1.	$\neg q \rightarrow \neg p$	Direct Proof Rule
2.	$p \rightarrow q$	Contrapositive: 1

Proof by Contradiction: One way to prove $\neg p$

If we assume p and derive F (a contradiction), then we have proven $\neg p$.

1.1. p Assumption

...

1.3. F

1. $p \rightarrow F$ Direct Proof rule
2. $\neg p \vee F$ Law of Implication: 1
3. $\neg p$ Identity: 2

eg. to prove $p \rightarrow q$ Assume $\neg(p \rightarrow q) \equiv \neg p \wedge q$ and derive false

Even and Odd

Predicate Definitions
Even(x) $\equiv \exists y\ (x = 2y)$
Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "No integer is both even and odd."

English proof: $\neg \exists x\ (\text{Even}(x) \wedge \text{Odd}(x))$

$\equiv \forall x\ \neg(\text{Even}(x) \wedge \text{Odd}(x))$

Proof: Let a be an arbitrary integer.
We prove by contradiction
Suppose that a is both even and odd (*)
∴ $a = 2b$ for some integer b depending on a.
and $a = 2c+1$ for integer c depending on a
∴ $2b = 2c+1$ so $b = c + 1/2$
Contradiction since no two integers differ by 1/2.
∴ (*) is false. and so a cannot be both even and odd.

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove: "No integer is both even and odd."

English proof: $\neg \exists x\, (\text{Even}(x) \wedge \text{Odd}(x))$

$\equiv \forall x\, \neg(\text{Even}(x) \wedge \text{Odd}(x))$

Proof: We work by contradiction. Let x be an arbitrary integer and suppose that it is both even and odd. Then x=2a for some integer a and x=2b+1 for some integer b. Therefore 2a=2b+1 and hence a=b+½.

But two integers cannot differ by ½ so this is a contradiction. So, no integer is both even and odd. ■

Rational Numbers

Domain of Discourse
Real Numbers

- A real number x is *rational* iff there exist integers p and q with $q \neq 0$ such that $x = p/q$.

$$\text{Rational}(x) \equiv \exists p\, \exists q\ ((x = p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge q \neq 0)$$

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) ≡ $\exists p\ \exists q\ ((x = p/q) \wedge \text{Integer}(p) \wedge \text{Integer}(q) \wedge (q \neq 0))$

Prove: "If x and y are rational then xy is rational."

Rationality

Domain of Discourse
Real Numbers

Predicate Definitions
Rational(x) $\equiv \exists p\ \exists q\ ((x = p/q) \land \text{Integer}(p) \land \text{Integer}(q) \land (q \neq 0))$

Prove: "If x and y are rational then xy is rational."

Proof: Let x and y be rational numbers. Then, x = a/b for some integers a, b, where b≠0, and y = c/d for some integers c,d, where d≠0.

Multiplying, we get that xy = (ac)/(bd).

Since b and d are both non-zero, so is bd; furthermore, ac and bd are integers. It follows that xy is rational, by definition of rational. ∎

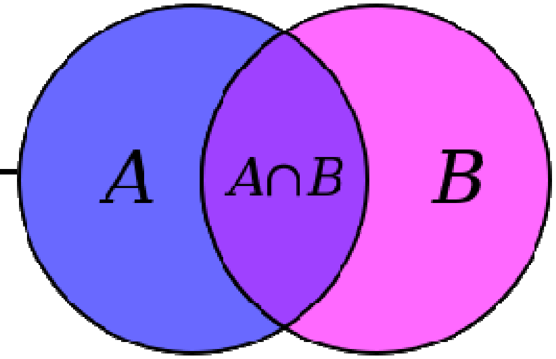
Proofs

- Formal proofs follow simple well-defined rules and should be easy to check
  - In the same way that code should be easy to execute

- English proofs correspond to those rules but are designed to be easier for humans to read
  - Easily checkable in principle

- Simple proof strategies already do a lot
  - Later we will cover a specific strategy that applies to loops and recursion (mathematical induction)

Set Theory



Sets are collections of objects called elements.

Write $a \in B$ **to say that** a **is an element of set** B**,**
and $a \notin B$ **to say that it is not.**

Some simple examples
A = {1}
B = {1, 3, 2}
C = {□, 1}
D = {{17}, 17}
E = {1, 2, 7, cat, dog, ∅, α}

Some Common Sets

$\mathbb{N}$ is the set of **Natural Numbers;** $\mathbb{N} = \{0, 1, 2, \ldots\}$

$\mathbb{Z}$ is the set of **Integers**; $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ is the set of **Rational Numbers**; e.g. ½, -17, 32/48

$\mathbb{R}$ is the set of **Real Numbers**; e.g. 1, -17, 32/48, $\pi, \sqrt{2}$

[n] is the set **{1, 2, …, n}** when **n** is a natural number

{} = $\varnothing$ is the **empty set**; the *only* set with no elements

Sets can be elements of other sets

For example

$A = \{\{1\},\{2\},\{1,2\},\emptyset\}$

$B = \{1,2\}$

Then $B \in A$.

Definitions

- A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\, (x \in A \leftrightarrow x \in B)$$

- A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\, (x \in A \rightarrow x \in B)$$

- Note: $(\boldsymbol{A} = \boldsymbol{B}) \equiv (\boldsymbol{A} \subseteq \boldsymbol{B}) \wedge (\boldsymbol{B} \subseteq \boldsymbol{A})$

Definition: Equality

A and B are *equal* if they have the same elements

$$A = B \equiv \forall x\,(x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

Which sets are equal to each other?

C = D = E

Definition: Subset

A is a *subset* of B if every element of A is also in B

$$A \subseteq B \equiv \forall x\,(x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}

QUESTIONS

$\varnothing \subseteq A$?

$A \subseteq B$?

$C \subseteq B$?

Building Sets from Predicates

S = the set of all* x for which P(x) is true

$$S = \{x : P(x)\}$$

S = the set of all x in A for which P(x) is true

$$S = \{x \in A : P(x)\}$$

*in the domain of P, usually called the "universe" U

Set Operations

$$A \cup B = \{ x : (x \in A) \vee (x \in B)\}$$ **Union**

$$A \cap B = \{ x : (x \in A) \wedge (x \in B)\}$$ **Intersection**

$$A \setminus B = \{ x : (x \in A) \wedge (x \notin B)\}$$ **Set Difference**

A = {1, 2, 3}
B = {3, 5, 6}
C = {3, 4}

QUESTIONS

Using A, B, C and set operations, make…

[6] = $A \cup B \cup C$

{3} = $B \cap C = A \cap B$

{1,2} = $A \setminus C = A \setminus B$

More Set Operations

$$A \oplus B = \{\, x : (x \in A) \oplus (x \in B)\}$$

Symmetric Difference

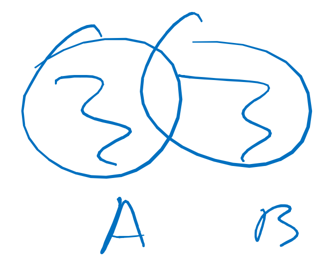
$$\overline{A} = \{\, x : x \notin A \,\} = \{ x \mid \neg(x \in A) \}$$

(with respect to universe U)

Complement

A = {1, 2, 3}
B = {1, 2, 4, 6}
Universe:
U = {1, 2, 3, 4, 5, 6}

$A \oplus B$ = {3, 4, 6}
$\overline{A}$ = {4,5,6}



It's Boolean algebra again

- Definition for $\cup$ based on $\vee$

- Definition for $\cap$ based on $\wedge$

- Complement works like $\neg$

De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$\neg$ $\{x : \neg((x \in A) \vee (x \in B))\}$

$= \{x : \neg(x \in A) \wedge \neg(x \in B)\}$

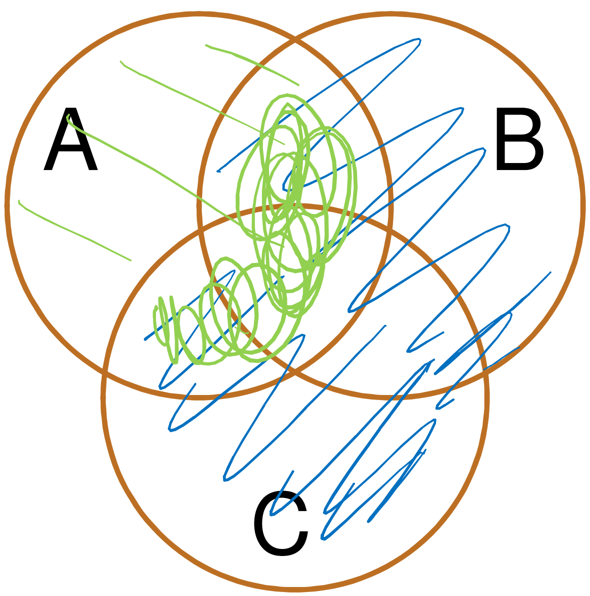
$= \{x : x \in \overline{A} \wedge x \in \overline{B}\}$

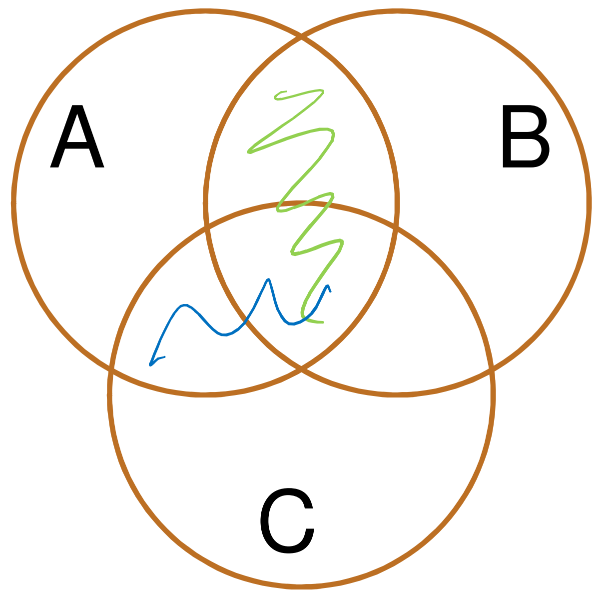
$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Proof technique:
To show C = D show
$x \in$ C $\rightarrow$ $x \in$ D and
$x \in$ D $\rightarrow$ $x \in$ C

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$





Power Set

- **Power Set of a set A = set of all subsets of A**

$$\mathcal{P}(A) = \{ B : B \subseteq A \}$$

- **e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class**

$\mathcal{P}$(Days)=?

$\mathcal{P}(\varnothing)$=?

Power Set

- **Power Set of a set A = set of all subsets of A**

$$\mathcal{P}(A) = \{\, B : B \subseteq A \,\}$$

- **e.g., let Days={M,W,F} and consider all the possible sets of days in a week you could ask a question in class**

$$\mathcal{P}(\text{Days})=\{\{M, W, F\}, \{M, W\}, \{M, F\}, \{W, F\}, \{M\}, \{W\}, \{F\}, \varnothing\}$$

$$\mathcal{P}(\varnothing)=\{\varnothing\} \neq \varnothing$$

Cartesian Product

$$A \times B = \{\,(a, b) : a \in A, b \in B\,\}$$

$\mathbb{R} \times \mathbb{R}$ is the real plane. You've seen ordered pairs before.

These are just for arbitrary sets.

$\mathbb{Z} \times \mathbb{Z}$ is "the set of all pairs of integers"

If A = {1, 2}, B = {a, b, c}, then A × B = {(1,a), (1,b), (1,c),
(2,a), (2,b), (2,c)}.

$\boldsymbol{A} \times \emptyset = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \wedge \boldsymbol{b} \in \emptyset\} = \{(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \boldsymbol{A} \wedge \mathbf{F}\} = \emptyset$

Representing Sets Using Bits

- Suppose universe U is $\{1,2, \ldots, n\}$
- Can represent set $B \subseteq U$ as a vector of bits:

  $b_1 b_2 \ldots b_n$ where $b_i = 1$ when $i \in B$

  $b_i = 0$ when $i \notin B$

  – Called the *characteristic vector* of set B

- Given characteristic vectors for A and B
  – What is characteristic vector for $A \cup B$? $A \cap B$?

UNIX/Linux File Permissions

- **`ls -l`**

```
drwxr-xr-x ... Documents/
-rw-r--r-- ... file1
```

- Permissions maintained as bit vectors
  - Letter means bit is 1
  - "–" means bit is 0.

Bitwise Operations

$$\begin{array}{rl} & 01101101 \\ \vee & \underline{00110111} \\ & 01111111 \end{array}$$

Java: `z=x|y`

$$\begin{array}{rl} & 00101010 \\ \wedge & \underline{00001111} \\ & 00001010 \end{array}$$

Java: `z=x&y`

$$\begin{array}{rl} & 01101101 \\ \oplus & \underline{00110111} \\ & 01011010 \end{array}$$

Java: `z=x^y`

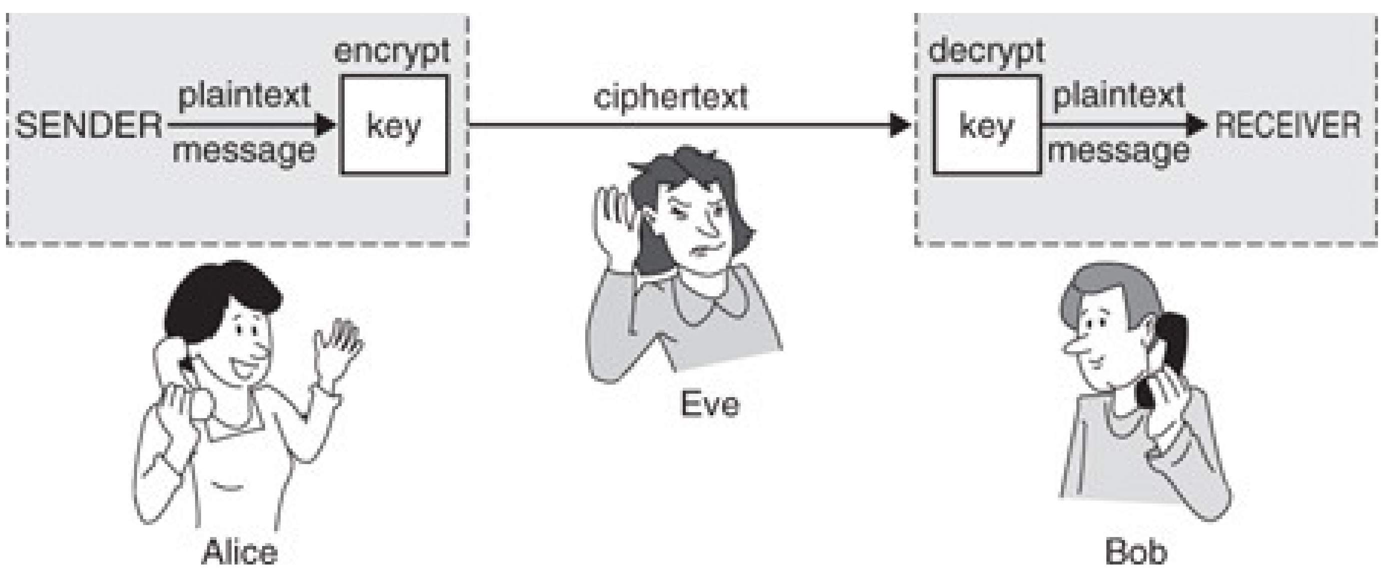
A Useful Identity

- If x and y are bits: $(x \oplus y) \oplus y = ?$

- What if x and y are bit-vectors?

Private Key Cryptography

- Alice wants to communicate message secretly to Bob so that eavesdropper Eve who hears their conversation cannot tell what Alice's message is.
- Alice and Bob can get together and privately share a secret key K ahead of time.



One-Time Pad

- Alice and Bob privately share random n-bit vector K
  - Eve does not know K
- Later, Alice has n-bit message m to send to Bob
  - Alice computes $C = m \oplus K$
  - Alice sends C to Bob
  - Bob computes $m = C \oplus K$ which is $(m \oplus K) \oplus K$
- Eve cannot figure out m from C unless she can guess K

Russell's Paradox

$$S = \{ x : x \notin x \}$$

Suppose for contradiction that $S \in S$...

Russell's Paradox

$$S = \{ x : x \notin x \}$$

Suppose for contradiction that $S \in S$. Then, by definition of S, $S \notin S$, but that's a contradiction.

Suppose for contradiction that $S \notin S$. Then, by definition of the set S, $S \in S$, but that's a contradiction, too.

This is reminiscent of the truth value of the statement "This statement is false."