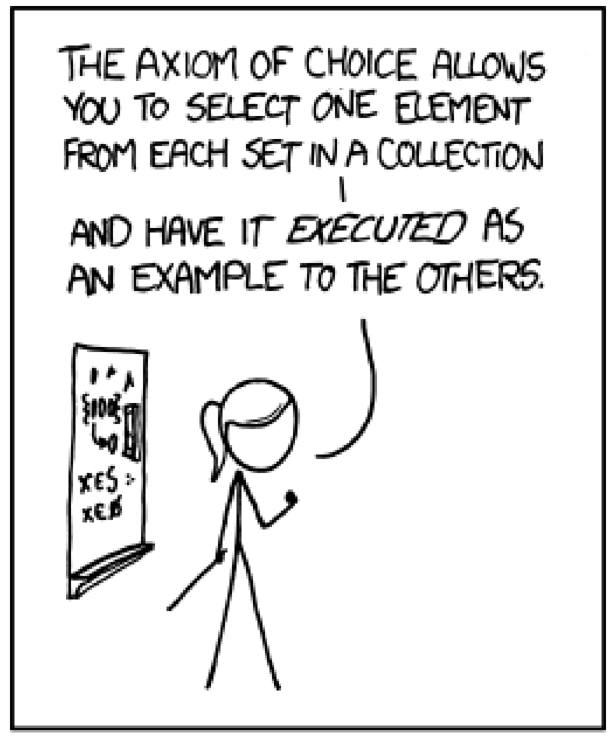
CSE 311: Foundations of Computing

Lecture 8: Predicate Logic Proofs



MY MATH TEACHER WAS A BIG BELIEVER IN PROOF BY INTIMIDATION.

Last class: Propositional Inference Rules

Two inference rules per binary connective, one to eliminate it and one to introduce it

Elim ∧: $\dfrac{A \land B}{\therefore A, B}$

Intro ∧: $\dfrac{A \;;\; B}{\therefore A \land B}$

Elim ∨: $\dfrac{A \lor B \;;\; \neg A}{\therefore B}$

Intro ∨: $\dfrac{A}{\therefore A \lor B,\ B \lor A}$

Modus Ponens: $\dfrac{A \;;\; A \rightarrow B}{\therefore B}$

Direct Proof Rule: $\dfrac{A \Rightarrow B}{\therefore A \rightarrow B}$

Not like other rules

Last class: Example

Prove: $((p \to q) \land (q \to r)) \to (p \to r)$

1.1.	$(p \to q) \land (q \to r)$	Assumption
1.2.	$p \to q$	∧ Elim: 1.1
1.3.	$q \to r$	∧ Elim: 1.1
1.4.1.	p	Assumption
1.4.2.	q	MP: 1.2, 1.4.1
1.4.3.	r	MP: 1.3, 1.4.2
1.4.	$p \to r$	Direct Proof Rule
1.	$((p \to q) \land (q \to r)) \to (p \to r)$	Direct Proof Rule

Inference Rules for Quantifiers: First look

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore \quad P(a) \textbf{ for any } a}$$

Intro ∀

$$\frac{\text{"}\textbf{Let } a \textbf{ be arbitrary}^{*}\text{"} \ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

* in the domain of P

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \textbf{ for some } \textbf{\textit{special}}^{**}\ c}$$

** By special, we mean that c is a name for a value where P(c) is true. We can't use anything else about that value, so c has to be a NEW name!

Predicate Logic Proofs

- Can use
  - Predicate logic inference rules

    whole formulas only
  - Predicate logic equivalences (De Morgan's)

    even on subformulas
  - Propositional logic inference rules

    whole formulas only
  - Propositional logic equivalences

    even on subformulas

My First Predicate Logic Proof

Intro ∃ $\dfrac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$

Elim ∀ $\dfrac{\forall x\, P(x)}{\therefore\ P(a) \text{ for any } a}$

Prove $\forall x\, P(x) \to \exists x\, P(x)$

$\forall x \;\; P(x)$

$\exists x \;\; P(x)$

5. $\forall x\, P(x) \to \exists x\, P(x)$ (?)

The main connective is implication so Direct Proof Rule seems good

My First Predicate Logic Proof

Intro ∃ $\dfrac{P(c) \text{ for some } c}{\therefore\ \exists x\, P(x)}$

Elim ∀ $\dfrac{\forall x\, P(x)}{\therefore\ P(a) \text{ for any } a}$

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

We need an ∃ we don't have so "intro ∃" rule makes sense

1.5. $\exists x\, P(x)$? intro ∃

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

We need an ∃ we don't have so "intro ∃" rule makes sense

1.5. $\exists x\, P(x)$ Intro ∃: ?

That requires P(c) for some c.

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃: $\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$

Elim ∀: $\frac{\forall x\, P(x)}{\therefore \ P(a) \text{ for any } a}$

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1. $\forall x\, P(x)$ Assumption

1.2 $P(a)$ Elim ∀: 1.1

We could have picked any name or domain expression here.

1.5. $\exists x\, P(x)$ Intro ∃: ?

That requires P(c) for some c.

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃: P(c) for some c ∴ ∃x P(x)

Elim ∀: ∀x P(x) ∴ P(a) for any a

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

No holes. Just need to clean up.

1.1. $\forall x\, P(x)$ Assumption

1.2 $P(a)$ Elim ∀: 1.1

1.5. $\exists x\, P(x)$ Intro ∃: 1.2

1. $\forall x\, P(x) \rightarrow \exists x\, P(x)$ Direct Proof Rule

My First Predicate Logic Proof

Intro ∃	P(c) for some c	
	∴ ∃x P(x)	

Elim ∀	∀x P(x)	
	∴ P(a) for any a	

Prove $\forall x\, P(x) \rightarrow \exists x\, P(x)$

1.1.	$\forall x\, P(x)$	Assumption
1.2	$P(a)$	Elim ∀: 1.1
1.3.	$\exists x\, P(x)$	Intro ∃: 1.2
1.	$\forall x\, P(x) \rightarrow \exists x\, P(x)$	Direct Proof Rule

Working forwards as well as backwards:
In applying "Intro ∃" rule we didn't know what expression we might be able to prove P(c) for, so we worked forwards to figure out what might work.

Predicate Logic Proofs with more content

- In propositional logic we could just write down other propositional logic statements as “givens”

- Here, we also want to be able to use domain knowledge so proofs are about something specific

- Example:

Domain of Discourse
Integers

- Given the basic properties of arithmetic on integers, define:

Predicate Definitions
$\text{Even}(x) \equiv \exists y\ (x = 2 \cdot y)$
$\text{Odd}(x) \equiv \exists y\ (x = 2 \cdot y + 1)$

A Not so Odd Example

Domain of Discourse
Integers

Predicate Definitions
$Even(x) \equiv \exists y\ (x = 2 \cdot y)$
$Odd(x) \equiv \exists y\ (x = 2 \cdot y + 1)$

Prove "There is an even number"

Formally: prove $\exists x\ Even(x)$

1. $2 = 2 \cdot 1$ ALGEBRA
2. $\exists y\ 2 = 2 \cdot y$ Intro $\exists$
3. Even(2)
4. $\exists x\ Even(x)$ Intro $\exists$

A Not so Odd Example

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) \equiv \exists y\ (x = 2 \cdot y)$
$\text{Odd}(x) \equiv \exists y\ (x = 2 \cdot y + 1)$

Prove "There is an even number"

Formally: prove $\exists x\ \text{Even}(x)$

1.	$\mathbf{2 = 2 \cdot 1}$	**Arithmetic**
2.	$\exists y\ (\mathbf{2} = \mathbf{2} \cdot y)$	**Intro** $\exists$**: 1**
3.	$\text{Even}(\mathbf{2})$	**Definition of** Even**: 2**
4.	$\exists x\ \text{Even}(x)$	**Intro** $\exists$**: 3**

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
$\text{Even}(x) \equiv \exists y\ (x = 2 \cdot y)$
$\text{Odd}(x) \equiv \exists y\ (x = 2 \cdot y + 1)$
Prime(x) ≡ "x > 1 and x≠a·b for all integers a, b with 1<a<x"

Prove "There is an even prime number"

$\exists x\ \text{Even}(x) \wedge \text{Prime}(x)$

1. $2 = 2 \cdot 1$
2. $\exists y\ 2 = 2 \cdot y$ — Intr ∃
3. Even(2) — Def Even
4. Prime(2) — Def prime
5. $\text{Even}(2) \wedge \text{Prime}(2)$ — Intro ∧
6. $\exists x\ \text{Even}(x) \wedge \text{Prime}(x)$ — Intro ∃

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) ≡ ∃y (x = 2·y) Odd(x) ≡ ∃y (x = 2·y + 1) Prime(x) ≡ "x > 1 and x≠a·b for all integers a, b with 1<a<x"

Prove "There is an even prime number"

Formally: prove ∃x (Even(x) ∧ Prime(x))

1.	**2 = 2·1**	**Arithmetic**
2.	Prime(**2**)*	**Property of integers**

*** Later we will further break down** "Prime" **using quantifiers to prove statements like this**

A Prime Example

Domain of Discourse
Integers

Predicate Definitions
Even(x) ≡ ∃y (x = 2·y)
Odd(x) ≡ ∃y (x = 2·y + 1)
Prime(x) ≡ "x > 1 and x≠a·b for all integers a, b with 1<a<x"

Prove "There is an even prime number"

Formally: prove ∃x (Even(x) ∧ Prime(x))

1.	**2 = 2·1**	**Arithmetic**
2.	Prime(**2**)*	**Property of integers**
3.	∃y (**2** = **2**·y)	**Intro** ∃**: 1**
4.	Even(**2**)	**Defn of** Even**: 3**
5.	Even(**2**) ∧ Prime(**2**)	**Intro** ∧**: 2, 4**
6.	∃x (Even(x) ∧ Prime(x))	**Intro** ∃**: 5**

*** Later we will further break down** "Prime" **using quantifiers to prove statements like this**

Inference Rules for Quantifiers: First look

Intro ∃

$$\frac{P(c) \text{ for some } c}{\therefore \quad \exists x\, P(x)}$$

Elim ∀

$$\frac{\forall x\, P(x)}{\therefore \quad P(a) \textbf{ for any } a}$$

Intro ∀

$$\frac{\text{"}\textbf{Let } a \textbf{ be arbitrary}^{*}\text{"} \ldots P(a)}{\therefore \quad \forall x\, P(x)}$$

* in the domain of P

Elim ∃

$$\frac{\exists x\, P(x)}{\therefore P(c) \textbf{ for some } \textbf{\textit{special}}^{**}\ c}$$

** By special, we mean that c is a name for a value where P(c) is true. We can't use anything else about that value, so c has to be a NEW name!

Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Intro ∀	"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	Elim ∃	∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let a be arbitrary

   2.1 Even(a) Assump

   2.10 Even(a^2)

2. Even(a) → Even(a^2) DPR

3. $\forall x\,(\text{Even}(x)\rightarrow\text{Even}(x^2))$? intro ∀

Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Intro ∀: "Let a be arbitrary*"...P(a) ∴ $\forall x\, P(x)$	Elim ∃: $\exists x\, P(x)$ ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

2. $\text{Even}(\mathbf{a}) \rightarrow \text{Even}(\mathbf{a}^2)$?
3. $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$ Intro ∀: 1,2

Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

1. Let a be an arbitrary integer
2.1 Even(a) Assumption
2.6 Even(a²)
2. Even(a)→Even(a²) Direct proof rule
3. ∀x (Even(x)→Even(x²)) Intro ∀: 1,2

Elim ∃

$$\frac{\exists x\ P(x)}{\therefore P(c) \text{ for some } \textbf{\textit{special}}^{**}\ c}$$

Prove: "The square of every even number is even."

Formal proof of: ∀x (Even(x) → Even(x²))

1. Let **a** be an arbitrary integer
   - 2.1 Even(**a**) Assumption
   - 2.2 ∃y a = 2·y Def of Ewn
   - 2.5 ∃y a² = 2·y
   - **2.6** Even(**a**²) ?
2. Even(**a**)→Even(**a**²) Direct proof rule
3. ∀x (Even(x)→Even(x²)) Intro ∀: 1,2

Even(x) ≡ $\exists y\ (x=2y)$
Odd(x) ≡ $\exists y\ (x=2y+1)$
Domain: Integers

Even and Odd

Intro ∀	Elim ∃
"Let a be arbitrary*"...P(a) / ∴ ∀x P(x)	∃x P(x) / ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer

    2.1 Even(**a**) — Assumption

    2.2 $\exists y\ (\mathbf{a} = 2y)$ — Definition of Even

    2.3 a = 2·y .

    2.5 $\exists y\ (\mathbf{a}^2 = 2y)$ — ? intro ∃

    2.6 Even($\mathbf{a}^2$) — Definition of Even

2. Even(**a**)→Even($\mathbf{a}^2$) — Direct proof rule

3. $\forall x\ (\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro ∀: 1,2

Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Intro ∀	Elim ∃
"Let a be arbitrary*"...P(a) ∴ ∀x P(x)	∃x P(x) ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: ∀x (Even(x) → Even(x^2))

1. Let **a** be an arbitrary integer
   - 2.1 Even(**a**) — Assumption
   - 2.2 ∃y (**a** = 2y) — Definition of Even
   - 2.5 ∃y ($\mathbf{a}^2$ = 2y) — Intro ∃ rule: ?
   - 2.6 Even($\mathbf{a}^2$) — Definition of Even
2. Even(**a**)→Even($\mathbf{a}^2$) — Direct proof rule
3. ∀x (Even(x)→Even(x^2)) — Intro ∀: 1,2

Need $\mathbf{a}^2$ = 2c for some **c**

Even and Odd

Even(x) ≡ ∃y (x=2y)
Odd(x) ≡ ∃y (x=2y+1)
Domain: Integers

Intro ∀	Elim ∃
"Let a be arbitrary*"...P(a) / ∴ ∀x P(x)	∃x P(x) / ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: ∀x (Even(x) → Even(x^2))

1. Let **a** be an arbitrary integer
   - 2.1 Even(**a**) — Assumption
   - 2.2 ∃y (**a** = 2y) — Definition of Even
   - 2.3 **a** = 2**b** — Elim ∃: **b** special depends on **a**
   - 2.4 $a^2 = 4b^2 = 2(2b^2)$
   - 2.5 ∃y (**a**^{2} = 2y) — Intro ∃ rule: ?
   - 2.6 Even(**a**^{2}) — Definition of Even
2. Even(**a**)→Even(**a**^{2}) — Direct proof rule
3. ∀x (Even(x)→Even(x^2)) — Intro ∀: 1,2

Need **a**^{2} = 2c for some **c**

Even and Odd

Even(x) $\equiv \exists y\ (x=2y)$
Odd(x) $\equiv \exists y\ (x=2y+1)$
Domain: Integers

Intro ∀	Elim ∃
"Let a be arbitrary*"…P(a) / ∴ $\forall x\, P(x)$	$\exists x\, P(x)$ / ∴ P(c) for some *special*** c

Prove: "The square of every even number is even."

Formal proof of: $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$

1. Let **a** be an arbitrary integer
   - 2.1 Even(**a**) — Assumption
   - 2.2 $\exists y\,(\mathbf{a} = 2y)$ — Definition of Even
   - 2.3 $\mathbf{a} = 2\mathbf{b}$ — Elim ∃: **b** special depends on **a**
   - 2.4 $\mathbf{a}^2 = 4\mathbf{b}^2 = 2(2\mathbf{b}^2)$ — Algebra
   - 2.5 $\exists y\,(\mathbf{a}^2 = 2y)$ — Intro ∃ rule
   - 2.6 Even($\mathbf{a}^2$) — Definition of Even
2. $\text{Even}(\mathbf{a}) \rightarrow \text{Even}(\mathbf{a}^2)$ — Direct proof rule
3. $\forall x\,(\text{Even}(x) \rightarrow \text{Even}(x^2))$ — Intro ∀: 1,2

Used $\mathbf{a}^2 = 2c$ for $c=2\mathbf{b}^2$

Why did we need to say that b depends on a?

There are extra conditions on using these rules:

Intro $\forall$: "Let a be arbitrary*"...P(a) ∴ $\forall x\, P(x)$

* in the domain of P

Elim $\exists$: $\exists x\, P(x)$ ∴ P(c) **for some *special***** c

** c has to be a NEW name.

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

BAD "PROOF"

1.	$\forall x\, \exists y\, (y \geq x)$	Given
2.	**Let a be an arbitrary integer**	
3.	$\exists y\, (y \geq \mathbf{a})$	Elim $\forall$: **1**
4.	$\mathbf{b} \geq \mathbf{a}$	Elim $\exists$: **b** special depends on **a**
5.	$\forall x\, (\mathbf{b} \geq x)$	Intro $\forall$: **2,4**
6.	$\exists y \forall x\, (y \geq x)$	Intro $\exists$: **5**

Why did we need to say that b depends on a?

There are extra conditions on using these rules:

Intro $\forall$: "Let a be arbitrary*"...P(a) ∴ $\forall x\, P(x)$

* in the domain of P

Elim $\exists$: $\exists x\, P(x)$ ∴ P(c) **for some *special***** c

** c has to be a NEW name.

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

BAD "PROOF"

1. $\forall x\, \exists y\, (y \geq x)$ — Given
2. **Let a be an arbitrary integer**
3. $\exists y\, (y \geq a)$ — Elim $\forall$: **1**
4. $b \geq a$ — Elim $\exists$: **b** special depends on **a**
5. $\forall x\, (b \geq x)$ — Intro $\forall$: **2,4**
6. $\exists y \forall x\, (y \geq x)$ — Intro $\exists$: **5**

Can't get rid of a since another name in the same line, b, depends on it!

Why did we need to say that b depends on a?

There are extra conditions on using these rules:

Intro ∀: "Let a be arbitrary*" ...P(a) ∴ ∀x P(x)

Elim ∃: ∃x P(x) ∴ P(c) for some *special*** c

* in the domain of P. No other name in P depends on a

** c is a NEW name. List all dependencies for c.

Over integer domain: $\forall x\, \exists y\, (y \geq x)$ **is True but** $\exists y \forall x\, (y \geq x)$ **is False**

BAD "PROOF"

1. $\forall x\, \exists y\, (y \geq x)$ — Given
2. Let a be an arbitrary integer
3. $\exists y\, (y \geq a)$ — Elim ∀: 1
4. $b \geq a$ — Elim ∃: b special depends on a
5. ~~$\forall x\, (b \geq x)$ — Intro ∀: 2,4~~
6. $\exists y \forall x\, (y \geq x)$ — Intro ∃ : 5

Can't get rid of a since another name in the same line, b, depends on it!

Inference Rules for Quantifiers: Full version

Intro ∃

P(c) for some c

∴ ∃x P(x)

Elim ∀

∀x P(x)

∴ P(a) **for any** a

Intro ∀

"**Let** a **be arbitrary***" ...P(a)

∴ ∀x P(x)

* in the domain of P. No other name in P depends on a

Elim ∃

∃x P(x)

∴ P(c) **for some *special***** c

** c is a NEW name.
List all dependencies for c.

English Proofs

- We often write proofs in English rather than as fully formal proofs
  - They are more natural to read

- English proofs follow the structure of the corresponding formal proofs
  - Formal proof methods help to understand how proofs really work in English…

    … and give clues for how to produce them.

An English Proof

Predicate Definitions

$Even(x) \equiv \exists y\ (x = 2 \cdot y)$

$Odd(x) \equiv \exists y\ (x = 2 \cdot y + 1)$

Prove "There is an even integer"

Proof:

$2 = 2 \cdot 1$	1.	$2 = 2 \cdot 1$	Arithmetic
so 2 equals 2 times an integer.	2.	$\exists y\ (2 = 2 \cdot y)$	Intro $\exists$: 1
Therefore 2 is even.	3.	$Even(2)$	Defn of Even: 2
Therefore, there is an even integer ∎	4.	$\exists x\ Even(x)$	Intro $\exists$: 3

English Even and Odd

Even(x) ≡ $\exists y\ (x=2y)$
Odd(x) ≡ $\exists y\ (x=2y+1)$
Domain: Integers

Prove "The square of every even integer is even."

Proof: Let a be an arbitrary even integer.	1. Let a be an arbitrary integer 2.1 Even(a) — Assumption
Then, by definition, $a = 2b$ for some integer b (depending on a).	2.2 $\exists y\ (a = 2y)$ — Definition 2.3 $a = 2b$ — b special depends on a
Squaring both sides, we get $a^2 = 4b^2 = 2(2b^2)$.	2.4 $a^2 = 4b^2 = 2(2b^2)$ — Algebra
Since $2b^2$ is an integer, by definition, a^2 is even.	2.5 $\exists y\ (a^2 = 2y)$ 2.6 Even(a^2) — Definition
Since a was arbitrary, it follows that the square of every even number is even. ■	2. Even(a)→Even(a^2) 3. $\forall x\ (\text{Even}(x) \to \text{Even}(x^2))$

Even and Odd

Predicate Definitions

Even(x) $\equiv \exists y\ (x = 2y)$

Odd(x) $\equiv \exists y\ (x = 2y + 1)$

Domain of Discourse

Integers

Prove "The square of every odd number is odd."

Even and Odd

Predicate Definitions
Even(x) ≡ $\exists y\ (x = 2y)$
Odd(x) ≡ $\exists y\ (x = 2y + 1)$

Domain of Discourse
Integers

Prove "The square of every odd number is odd."

Proof: Let b be an arbitrary odd number.

Then, b = 2c+1 for some integer c (depending on b).

Therefore, $b^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1$.

Since $2c^2+2c$ is an integer, b^2 is odd. The statement follows since b was arbitrary. ■

Proofs

- Formal proofs follow simple well-defined rules and should be easy to check
  - In the same way that code should be easy to execute

- English proofs correspond to those rules but are designed to be easier for humans to read
  - Easily checkable in principle