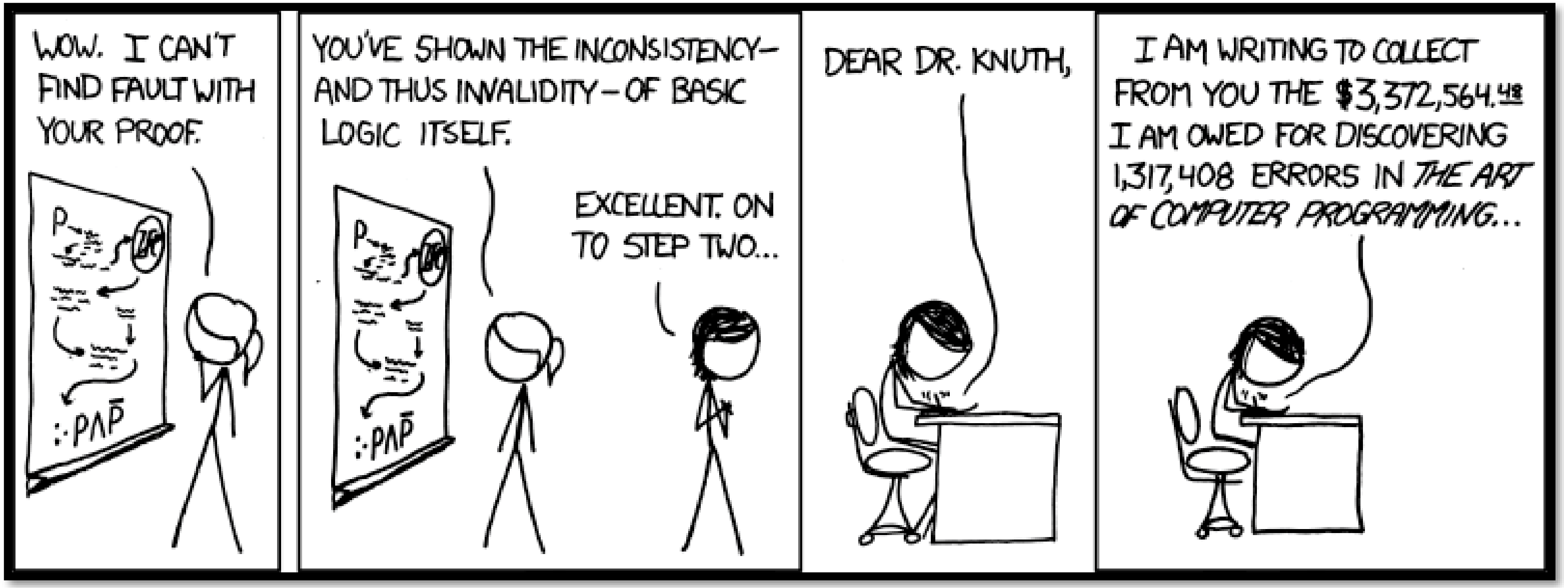
# CSE 311: Foundations of Computing

## Lecture 7: Logical Inference

Regrade Request:
Do it till Friday.



# Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

# Applications of Logical Inference

- Software Engineering
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- Artificial Intelligence
  - Automated reasoning
- Algorithm design and analysis
  - e.g., Correctness, Loop invariants.
- Logic Programming, e.g. Prolog
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

# Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

# An inference rule: *Modus Ponens*

- **If A and A → B are both true then B must be true**

- **Write this rule as**

$$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$$

- **Given:**

  $A \longrightarrow B$

  - **If it is Wednesday then you have a 311 class today.**
  - **It is Wednesday.**

  $A$

- **Therefore, by Modus Ponens:**
  - **You have a 311 class today.**

# My First Proof!

Show that r follows from p, p → q, and q → r

| | | |
|---|---|---|
| 1. | $p$ | Given |
| 2. | $p \rightarrow q$ | Given |
| 3. | $q \rightarrow r$ | Given |
| 4. | q | MP of 1, 2. |
| 5. | r | MP of 3,4 |

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

# My First Proof!

Show that $r$ follows from $p$, $p \to q$, and $q \to r$

| | | |
|---|---|---|
| 1. | $p$ | Given |
| 2. | $p \to q$ | Given |
| 3. | $q \to r$ | Given |
| 4. | $q$ | MP: 1, 2 |
| 5. | $r$ | MP: 3, 4 |

Modus Ponens


| | | |
|---|---|---|
| 1. | $p$ | Given |
| 2. | $p \to q$ | Given |
| 3. | $q \to r$ | Given |
| 4. | $q$ | MP: 1, 2 |
| 5. | $r$ | MP: 3, 4 |

# Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

| | | |
|---|---|---|
| 1. | $p \rightarrow q$ | Given |
| 2. | $\neg q$ | Given |
| 3. | $\neg q \rightarrow \neg p$ | Contrapositive: 1 |
| 4. | $\neg p$ | MP: 2, 3 |

Modus Ponens $\dfrac{A\ ;\ A \rightarrow B}{\therefore\ B}$

# Inference Rules

If A is true and B is true ....

Requirements: $A \;;\; B$

Conclusions: $\therefore\; C \;,\; D$

$$\frac{A \;;\; B}{\therefore\; C \;,\; D}$$

Then, C must be true

Then D must be true

Example (Modus Ponens):

$$\frac{A \;;\; A \rightarrow B}{\therefore\; B}$$

If I have A and $A \rightarrow B$ both true,
Then B must be true.

# Axioms: Special inference rules

If I have nothing...

Requirements:

Conclusions: $\therefore C \; , \; D$

Then, C must be true

Then D must be true

Example (Excluded Middle):

$$\frac{\qquad}{\therefore\ A \vee \neg A}$$

$A \vee \neg A$ must be true.

# Simple Propositional Inference Rules

**Two inference rules per binary connective, one to eliminate it and one to introduce it**

Elim $\wedge$
$$\frac{A \wedge B}{\therefore A, B}$$

Intro $\wedge$
$$\frac{A \; ; \; B}{\therefore A \wedge B}$$

Elim $\vee$
$$\frac{A \vee B \; ; \; \neg A}{\therefore B}$$

Intro $\vee$
$$\frac{A}{\therefore A \vee B, B \vee A}$$

Modus Ponens
$$\frac{A \; ; \; A \rightarrow B}{\therefore B}$$

Direct Proof Rule
$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Not like other rules

# Proofs

Show that r follows from $p$, $p \rightarrow q$ and $(p \wedge q) \rightarrow r$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{A\ ;\ A \rightarrow B}{\therefore\ B}$$

$$\frac{A \wedge B}{\therefore\ A,\ B}$$

$$\frac{A\ ;\ B}{\therefore\ A \wedge B}$$

1. $p$ Given
2. $p \rightarrow q$ Given
3. $(p \wedge q) \rightarrow r$ Given
4. $q$ MP of 1,2
5. $p \wedge q$ Intro $\wedge$ 1,4
6. $r$ MP of 3,5

# Proofs

Show that $r$ follows from $p, p \to q,$ and $p \wedge q \to r$

| | | |
|---|---|---|
| 1. | $p$ | Given |
| 2. | $p \to q$ | Given |
| 3. | $q$ | MP: 1, 2 |
| 4. | $p \wedge q$ | Intro $\wedge$: 1, 3 |
| 5. | $p \wedge q \to r$ | Given |
| 6. | $r$ | MP: 4, 5 |

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!

$$\dfrac{\dfrac{p \quad ; \quad \dfrac{p \quad ; \quad p \to q}{q}\,\text{MP}}{p \wedge q}\,\text{Intro } \wedge \qquad ; \qquad p \wedge q \to r}{r}\,\text{MP}$$

# Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \rightarrow r$ ✓ given

~~2. $(p \vee q) \rightarrow r$ intro $\vee$ from 1.~~

$\underbrace{(p \vee q)}_{T} \rightarrow \underbrace{r}_{F}$ X

Does not follow! e.g . p=**F**, q=**T**, r=**F**

# Proofs

Givens

Prove that ¬r follows from p ∧ s, q → ¬r, and ¬s ∨ q.

1. $p \wedge s$ Given
2. $q \rightarrow \neg r$ Given
3. $\neg s \vee q$ Given

**First: Write down givens and goal**

20. $\neg r$ ?

**Idea: Work backwards!**

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

1. $p \wedge s$ Given
2. $\boxed{q \rightarrow \neg r \quad \text{Given}}$
3. $\neg s \vee q$ Given

20. $\neg r$ MP: 2, ?

**Idea: Work backwards!**

We want to eventually get $\neg r$. How?

- We can use $q \rightarrow \neg r$ to get there.
- The justification between 2 and 20 looks like "elim $\rightarrow$" which is MP.

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \to \neg r$, and $\neg s \vee q$.

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \to \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 19. | $q$ | ? |
| 20. | $\neg r$ | MP: 2, 19 |

**Idea: Work backwards!**

We want to eventually get $\neg r$. How?

- Now, we have a new "hole"
- We need to prove $q$…
  - Notice that at this point, if we prove $q$, we've proven $\neg r$…

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \to \neg r$, and $\neg s \vee q$.

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \to \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 19. | $q$ | ? |
| 20. | $\neg r$ | MP: 2, 19 |

This looks like or-elimination.

Elim $\vee$ $\dfrac{A \vee B\ ;\ \neg A}{\therefore B}$

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \rightarrow \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 18. | $\neg\neg s$ | ? |
| 19. | $q$ | $\vee$ Elim: 3, 18 |
| 20. | $\neg r$ | MP: 2, 19 |

$\neg\neg s$ doesn't show up in the givens but $s$ does and we can use equivalences

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \rightarrow \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 17. | $s$ | ? |
| 18. | $\neg\neg s$ | Double Negation: 17 |
| 19. | $q$ | $\vee$ Elim: 3, 18 |
| 20. | $\neg r$ | MP: 2, 19 |

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \to \neg r$, and $\neg s \vee q$.

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \to \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| | | |
| 17. | $s$ | $\wedge$ Elim: 1 |
| 18. | $\neg\neg s$ | Double Negation: 17 |
| 19. | $q$ | $\vee$ Elim: 3, 18 |
| 20. | $\neg r$ | MP: 2, 19 |

No holes left! We just need to clean up a bit.

# Proofs

Prove that $\neg r$ follows from $p \wedge s$, $q \rightarrow \neg r$, and $\neg s \vee q$.

| | | |
|---|---|---|
| 1. | $p \wedge s$ | Given |
| 2. | $q \rightarrow \neg r$ | Given |
| 3. | $\neg s \vee q$ | Given |
| 4. | $s$ | $\wedge$ Elim: 1 |
| 5. | $\neg\neg s$ | Double Negation: 4 |
| 6. | $q$ | $\vee$ Elim: 3, 5 |
| 7. | $\neg r$ | MP: 2, 6 |

# To Prove An Implication: $A \rightarrow B$

- We use the direct proof rule
- The "pre-requisite" $A \Rightarrow B$ for the direct proof rule is **a proof** that "**Given** A, we can prove B."
- The direct proof rule:

  If you have such a proof then you can conclude that $A \rightarrow B$ is true

Example: Prove $p \rightarrow (p \vee q)$.

proof subroutine

Indent proof subroutine $\Rightarrow$

| | | |
|---|---|---|
| 1. | $p$ | Assumption |
| 2. | $p \vee q$ | Intro $\vee$: 1 |
| 3. | $p \rightarrow (p \vee q)$ | Direct Proof Rule |

# Proofs using the direct proof rule

Show that $p \to r$ follows from $q$ and $(p \land q) \to r$

This is a proof of $p \to r$

1. $q$ Given
2. $(p \land q) \to r$ Given

    3.1. $p$ Assumption

    3.2. $p \land q$ Intro $\land$: 1, 3.1

    3.3. $r$ MP: 2, 3.2

3. $\boxed{p \to r}$ Direct Proof Rule

If we know $p$ is true… Then, we've shown $r$ is true

# Example

Prove: $(p \wedge q) \rightarrow (p \vee q)$

There MUST be an application of the Direct Proof Rule (or an equivalence) to prove this implication.

Where do we start? We have no givens…

# Example

**Prove:** $(p \wedge q) \rightarrow (p \vee q)$

1.1 $p \wedge q$ Assumption
1.2 $p$ elim $\wedge$ in 1.1

1.10 $p \vee q$ Intro $\vee$ in 1.2
1 $(p \wedge q) \rightarrow (p \vee q)$ Direct Proof Rule.

# Example

**Prove:** $(p \wedge q) \rightarrow (p \vee q)$

| | | |
|---|---|---|
| 1.1. | $p \wedge q$ | Assumption |
| 1.2. | $p$ | Elim $\wedge$: 1.1 |
| 1.3. | $p \vee q$ | Intro $\vee$: 1.2 |
| 1. | $(p \wedge q) \rightarrow (p \vee q)$ | Direct Proof Rule |

# Example

**Prove:** $((p \to q) \wedge (q \to r)) \to (p \to r)$

1.1. $((p \to q) \wedge (q \to r))$ Assumption

1.2. $p \to q$ elim of $\wedge$ 1.1

1.3 $q \to r$ elim of $\wedge$ 1.1

1.20.1 $p$ Assumption

1.2.0.2 $q$ MP of 1.2, 1.20.1

1.20.20 $r$ MP 1.3 and 1.20.2

1.20. $p \to r$ DPR

1. $((p \to q) \wedge (q \to r)) \to (p \to r)$ DPR

# Example

**Prove:** $((p \to q) \land (q \to r)) \to (p \to r)$

| | | |
|---|---|---|
| 1.1. | $(p \to q) \land (q \to r)$ | Assumption |
| 1.2. | $p \to q$ | $\land$ Elim: 1.1 |
| 1.3. | $q \to r$ | $\land$ Elim: 1.1 |
| 1.4.1. | $p$ | Assumption |
| 1.4.2. | $q$ | MP: 1.2, 1.4.1 |
| 1.4.3. | $r$ | MP: 1.3, 1.4.2 |
| 1.4. | $p \to r$ | Direct Proof Rule |
| 1. | $((p \to q) \land (q \to r)) \to (p \to r)$ | Direct Proof Rule |

# One General Proof Strategy

1. Look at the rules for introducing connectives to see how you would build up the formula you want to prove from pieces of what is given

2. Use the rules for eliminating connectives to break down the given formulas so that you get the pieces you need to do 1.

3. Write the proof beginning with what you figured out for 2 followed by 1.