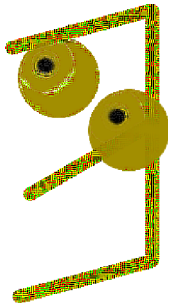
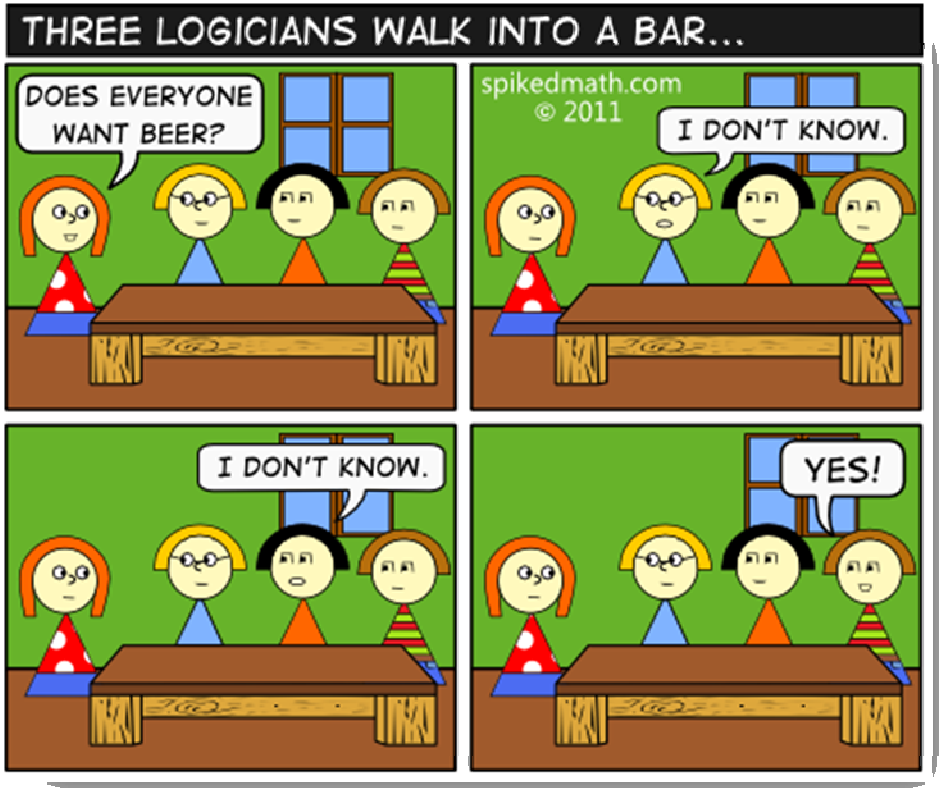
CSE 311: Foundations of Computing

Lecture 6: More Predicate Logic





Last class: Predicates

Predicate

– **A function that returns a truth value, e.g.,**

Cat(x) ::= “x is a cat”

Prime(x) ::= “x is prime”

HasTaken(x, y) ::= “student x has taken course y”

LessThan(x, y) ::= “x < y”

Sum(x, y, z) ::= “x + y = z”

GreaterThan5(x) ::= “x > 5”

HasNChars(s, n) ::= “string s has length n”

Predicates can have varying numbers of arguments and input types.

Last class: Domain of Discourse

For ease of use, we define one "type"/"domain" that we work over. This set of objects is called the "**domain of discourse**".

For each of the following, what might the domain be?

(1) "x is a cat", "x barks", "x ruined my couch"

(2) "x is prime", "x = 0", "x < 0", "x is a power of two"

(3) "x is a pre-req for z"

Domain of Discourse

For ease of use, we define one "type"/"domain" that we work over. This set of objects is called the "**domain of discourse**".

For each of the following, what might the domain be?

(1) "x is a cat", "x barks", "x ruined my couch"

"mammals" or "sentient beings" or "cats and dogs" or …

(2) "x is prime", "x = 0", "x < 0", "x is a power of two"

"numbers" or "integers" or "integers greater than 5" or …

(3) "x is a pre-req for z"

"courses"

Last Class: Quantifiers

We use ***quantifiers*** to talk about collections of objects.

$\forall x\ P(x)$

$P(x)$ is true **for every** x in the domain

read as "**for all x, P of x**"

$\exists x\ P(x)$

There is an x in the domain for which $P(x)$ is true

read as "**there exists x, P of x**"

Last class: Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= “x is even”	Greater(x, y) ::= “x > y”
Odd(x) ::= “x is odd”	Equal(x, y) ::= “x = y”
Prime(x) ::= “x is prime”	Sum(x, y, z) ::= “x + y = z”

Determine the truth values of each of these statements:

$\exists x$ Even(x)

$\forall x$ Odd(x)

$\forall x$ (Even(x) $\vee$ Odd(x))

$\exists x$ (Even(x) $\wedge$ Odd(x))

$\forall x$ Greater(x+1, x)

$\exists x$ (Even(x) $\wedge$ Prime(x))

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= “x is even”	Greater(x, y) ::= “x > y”
Odd(x) ::= “x is odd”	Equal(x, y) ::= “x = y”
Prime(x) ::= “x is prime”	Sum(x, y, z) ::= “x + y = z”

Determine the truth values of each of these statements:

$\exists x$ Even(x)	**T**	**e.g. 2, 4, 6, …**
$\forall x$ Odd(x)	**F**	**e.g. 2, 4, 6, …**
$\forall x$ (Even(x) $\vee$ Odd(x))	**T**	**every integer is either even or odd**
$\exists x$ (Even(x) $\wedge$ Odd(x))	**F**	**no integer is both even and odd**
$\forall x$ Greater(x+1, x)	**T**	**adding 1 makes a bigger number**
$\exists x$ (Even(x) $\wedge$ Prime(x))	**T**	**Even(2) is true and Prime(2) is true**

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= “x is even”	Greater(x, y) ::= “x > y”
Odd(x) ::= “x is odd”	Equal(x, y) ::= “x = y”
Prime(x) ::= “x is prime”	Sum(x, y, z) ::= “x + y = z”

Translate the following statements to English

$\forall x \, \exists y$ Greater(y, x)

$\forall x \, \exists y$ Greater(x, y)

$\forall x \, \exists y$ (Greater(y, x) $\wedge$ Prime(y))

$\forall x$ (Prime(x) $\rightarrow$ (Equal(x, 2) $\vee$ Odd(x)))

$\exists x \, \exists y$ (Sum(x, 2, y) $\wedge$ Prime(x) $\wedge$ Prime(y))

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= “x is even”	Greater(x, y) ::= “x > y”
Odd(x) ::= “x is odd”	Equal(x, y) ::= “x = y”
Prime(x) ::= “x is prime”	Sum(x, y, z) ::= “x + y = z”

Translate the following statements to English

$\forall x \exists y$ Greater(y, x)

For every positive integer x, there is a positive integer y, such that y > x.

$\forall x \exists y$ Greater(x, y)

For every positive integer x, there is a positive integer y, such that x > y.

$\forall x \exists y$ (Greater(y, x) $\wedge$ Prime(y))

For every positive integer x, there is a pos. int. y such that y > x and y is prime.

$\forall x$ (Prime(x) $\rightarrow$ (Equal(x, 2) $\vee$ Odd(x)))

For each positive integer x, if x is prime, then x = 2 or x is odd.

$\exists x \exists y$ (Sum(x, 2, y) $\wedge$ Prime(x) $\wedge$ Prime(y))

There exist positive integers x and y such that x + 2 = y and x and y are prime.

Statements with Quantifiers (Natural Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= “x is even”	Greater(x, y) ::= “x > y”
Odd(x) ::= “x is odd”	Equal(x, y) ::= “x = y”
Prime(x) ::= “x is prime”	Sum(x, y, z) ::= “x + y = z”

Translate the following statements to English

$\forall x \exists y$ Greater(y, x)

There is no greatest integer.

$\forall x \exists y$ Greater(x, y)

There is no least integer.

$\forall x \exists y$ (Greater(y, x) $\wedge$ Prime(y))

For every positive integer there is a larger number that is prime.

$\forall x$ (Prime(x) $\rightarrow$ (Equal(x, 2) $\vee$ Odd(x)))

Every prime number is either 2 or odd.

$\exists x \exists y$ (Sum(x, 2, y) $\wedge$ Prime(x) $\wedge$ Prime(y))

There exist prime numbers that differ by two.”

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= “x is a cat”
Red(x) ::= “x is red”
LikesTofu(x) ::= “x likes tofu”

“Red cats like tofu”

“Some red cats don’t like tofu”

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= “x is a cat”
Red(x) ::= “x is red”
LikesTofu(x) ::= “x likes tofu”

“Red cats like tofu”

$$\forall x ((\text{Red}(x) \wedge \text{Cat}(x)) \rightarrow \text{LikesTofu}(x))$$

“Some red cats don’t like tofu”

$$\exists y ((\text{Red}(y) \wedge \text{Cat}(y)) \wedge \neg \text{LikesTofu}(y))$$

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= “x is a cat”
Red(x) ::= “x is red”
LikesTofu(x) ::= “x likes tofu”

When putting two predicates together like this, we use an “and”.

“Red cats like tofu”

When restricting to a smaller domain in a “for all” we use **implication**.

When there’s no leading quantification, it means “for all”.

“Some red cats don’t like tofu”

When restricting to a smaller domain in an “exists” we use **and**.

“Some” means “there exists”.

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= “x is a purple fruit”

(*) $\forall$x PurpleFruit(x) (“All fruits are purple”)

What is the negation of (*)?

(a) “there exists a purple fruit”

(b) “there exists a non-purple fruit”

(c) “all fruits are not purple”

Try your intuition! Which one “feels” right?

Key Idea: In every domain, exactly one of a statement and its negation should be true.

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Key Idea: In every domain, exactly one of a statement and its negation should be true.

Domain of Discourse
{plum}

(*), (a)

Domain of Discourse
{apple}

(b), (c)

Domain of Discourse
{plum, apple}

(a), (b)

Negations of Quantifiers

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(*) $\forall$x PurpleFruit(x) (“All fruits are purple”)

What is the negation of (*)?

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(b) “there exists a non-purple fruit”
(c) “all fruits are not purple”

Key Idea: In every domain, exactly one of a statement and its negation should be true.

Domain of Discourse
{plum}

(*), (a)

Domain of Discourse
{apple}

(b), (c)

Domain of Discourse
{plum, apple}

(a), (b)

The only choice that ensures exactly one of the statement and its negation is (b).

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

De Morgan's Laws for Quantifiers

$$\neg\forall x\ P(x) \equiv \exists x \neg P(x)$$
$$\neg\exists x\ P(x) \equiv \forall x \neg P(x)$$

"There is no largest integer"

$$\begin{aligned}
&\neg \exists x \forall y\ (x \geq y) \\
\equiv\ &\forall x \neg \forall y\ (x \geq y) \\
\equiv\ &\forall x\ \exists y \neg (x \geq y) \\
\equiv\ &\forall x\ \exists y\ (y > x)
\end{aligned}$$

"For every integer there is a larger integer"

Scope of Quantifiers

$\exists x \ (P(x) \wedge Q(x))$ **vs.** $\exists x \, P(x) \wedge \exists x \, Q(x)$

scope of quantifiers

$\exists x \ (P(x) \wedge Q(x))$ **vs.** $\exists x \, P(x) \wedge \exists x \, Q(x)$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

Scope of Quantifiers

Example: NotLargest(x) $\equiv \exists$ y Greater (y, x)

$\equiv \exists$ z Greater (z, x)

truth value:

doesn't depend on y or z "bound variables"

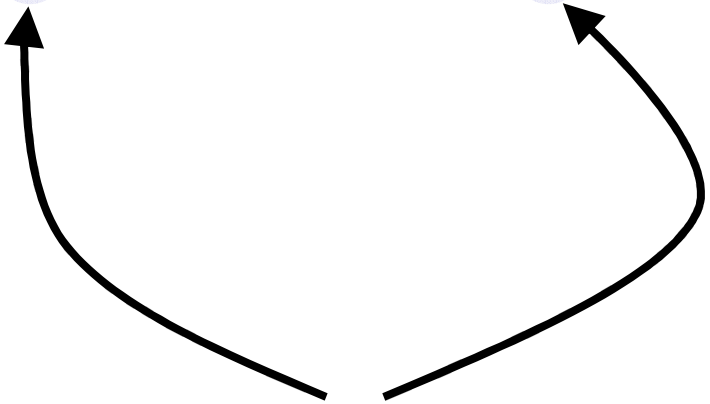
does depend on x "free variable"

quantifiers only act on free variables of the formula they quantify

$\forall$ x ($\exists$ y (P(x,y) $\rightarrow \forall$ x Q(y, x)))

Quantifier “Style”

$$\forall x(\exists y\ (P(x,y) \rightarrow \forall\ x\ Q(y,\ x)))$$



This isn’t “wrong”, it’s just horrible style.
Don’t confuse your reader by using the same variable multiple times…there are a lot of letters…

Nested Quantifiers

- **Bound variable names don't matter**

$$\forall x \exists y\ P(x, y) \equiv \forall a \exists b\ P(a, b)$$

- **Positions of quantifiers can sometimes change**

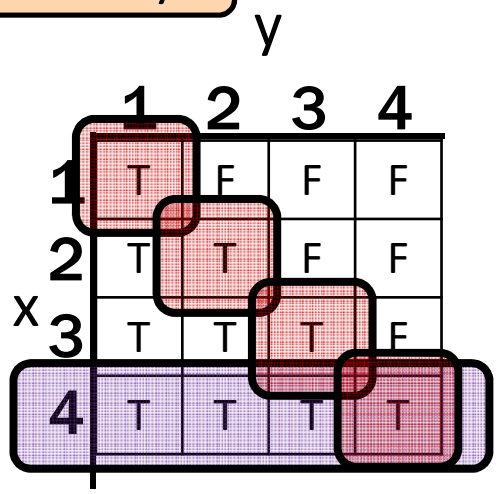
$$\forall x\ (Q(x) \wedge \exists y\ P(x, y)) \equiv \forall x \exists y\ (Q(x) \wedge P(x, y))$$

- **But: order is important…**

Quantifier Order Can Matter

Domain of Discourse
Integers OR {1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= “x ≥ y”



“There is a number greater than or equal to all numbers.”

$$\exists x \, \forall y \, \text{GreaterEq}(x, y)))$$

“Every number has a number greater than or equal to it.”

$$\forall y \, \exists x \, \text{GreaterEq}(x, y)))$$

The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Quantification with Two Variables

expression	when true	when false
$\forall x \forall y P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \forall x P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.

Logical Inference

- So far we've considered:
  - How to understand and *express* things using propositional and predicate logic
  - How to *compute* using Boolean (propositional) logic
  - How to show that different ways of expressing or computing them are *equivalent* to each other

- Logic also has methods that let us *infer* implied properties from ones that we know
  - Equivalence is a small part of this

Applications of Logical Inference

- **Software Engineering**
  - Express desired properties of program as set of logical constraints
  - Use inference rules to show that program implies that those constraints are satisfied
- **Artificial Intelligence**
  - Automated reasoning
- **Algorithm design and analysis**
  - e.g., Correctness, Loop invariants.
- **Logic Programming, e.g. Prolog**
  - Express desired outcome as set of constraints
  - Automatically apply logic inference to derive solution

Proofs

- Start with hypotheses and facts
- Use rules of inference to extend set of facts
- Result is proved when it is included in the set

An inference rule: *Modus Ponens*

- If p and p $\rightarrow$ q are both true then q must be true
- Write this rule as

$$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

- Given:
  - If it is Monday then you have a 311 class today.
  - It is Monday.
- Therefore, by Modus Ponens:
  - You have a 311 class today.

My First Proof!

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.		
5.		

My First Proof!

Show that r follows from p, $p \rightarrow q$, and $q \rightarrow r$

1.	p	Given
2.	$p \rightarrow q$	Given
3.	$q \rightarrow r$	Given
4.	q	MP: 1, 2
5.	r	MP: 3, 4

Proofs can use equivalences too

Show that $\neg p$ follows from $p \rightarrow q$ and $\neg q$

1.	$p \rightarrow q$	Given
2.	$\neg q$	Given
3.	$\neg q \rightarrow \neg p$	Contrapositive: 1
4.	$\neg p$	MP: 2, 3

Inference Rules

- Each inference rule is written as:
  ...which means that if both A and B are true then you can infer C and you can infer D.
  - For rule to be correct $(A \wedge B) \rightarrow C$ and $(A \wedge B) \rightarrow D$ must be a tautologies

$$\frac{A,\ B}{\therefore\ C,D}$$

- Sometimes rules don't need anything to start with. These rules are called axioms:
  - e.g. *Excluded Middle Axiom*

$$\frac{}{\therefore\ p \vee \neg p}$$

Simple Propositional Inference Rules

Excluded middle plus two inference rules per binary connective, one to eliminate it and one to introduce it

$$\frac{p \wedge q}{\therefore p, q} \qquad \frac{p, q}{\therefore p \wedge q}$$

$$\frac{p \vee q \ , \ \neg p}{\therefore q} \qquad \frac{p}{\therefore p \vee q, q \vee p}$$

$$\frac{p, \ p \rightarrow q}{\therefore \ q} \qquad \frac{p \Rightarrow q}{\therefore p \rightarrow q}$$

Direct Proof Rule

Not like other rules

Proofs

Show that r follows from p, $p \rightarrow q$ and $(p \wedge q) \rightarrow r$

How To Start:

We have givens, find the ones that go together and use them. Now, treat new things as givens, and repeat.

$$\frac{p,\ p \rightarrow q}{\therefore\ q}$$

$$\frac{p \wedge q}{\therefore\ p,\ q}$$

$$\frac{p,\ q}{\therefore\ p \wedge q}$$

Proofs

Show that r follows from $p, p \to q$, and $p \wedge q \to r$

Two visuals of the same proof. We will use the top one, but if the bottom one helps you think about it, that's great!

1.	p	Given
2.	$p \to q$	Given
3.	q	MP: 1, 2
4.	$p \wedge q$	Intro $\wedge$: 1, 3
5.	$p \wedge q \to r$	Given
6.	r	MP: 4, 5

$$\dfrac{\dfrac{p \quad p \to q}{q}\,\text{MP} \qquad p}{p \wedge q}\,\text{Intro } \wedge \qquad p \wedge q \to r$$

$$r\,\text{MP}$$

Important: Applications of Inference Rules

- You can use equivalences to make substitutions of any sub-formula.

- Inference rules only can be applied to whole formulas (not correct otherwise).

e.g. 1. $p \rightarrow q$ given

2. $(p \vee r) \rightarrow q$ intro $\vee$ from 1.

Does not follow! e.g . p=F, q=F, r=T