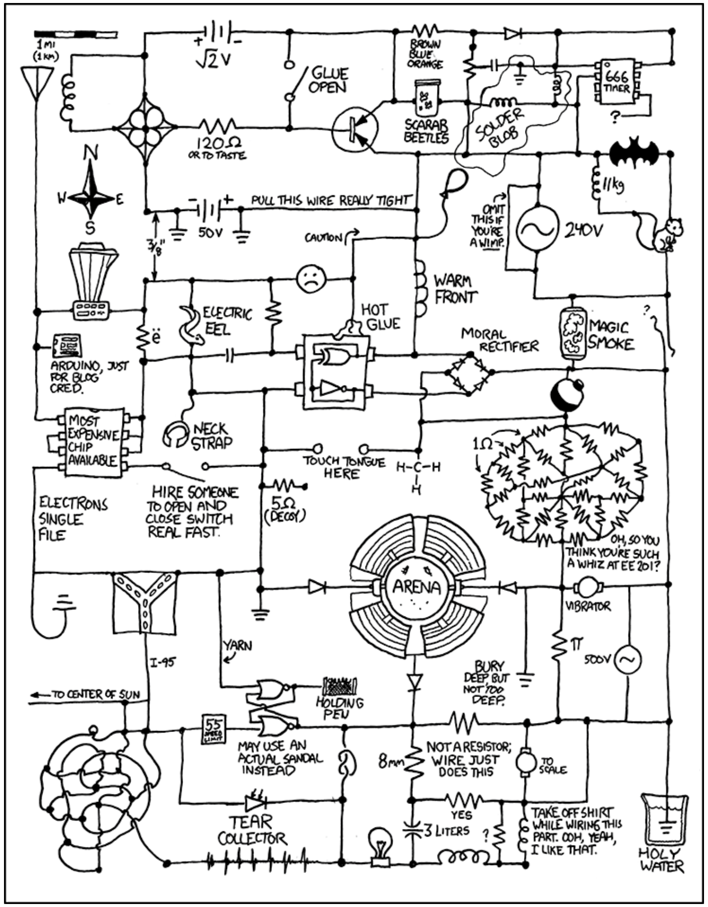
CSE 311: Foundations of Computing

Lecture 4: Boolean Algebra, Circuits, Canonical Forms

Monday 11:30 - 12:00

Paul: Tu 3:00 - 3:30



Boolean Logic

Combinational Logic

– output = F(input)

Sequential Logic

– $output_t = F(output_{t-1}, input_t)$
  - output dependent on history
  - concept of a time step (clock, t)

Boolean Algebra: Another notation for logic consisting of…

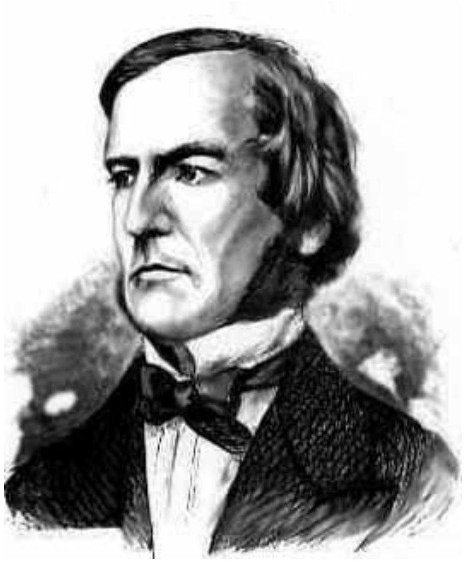
– a set of elements B = {0, 1}

– binary operations { + , • } (OR, AND)

– and a unary operation { ’ } (NOT)

$p \wedge q \quad p \cdot q$

Boolean Algebra



- Usual notation used in circuit design
- Boolean algebra
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { ’ }
  - such that the following axioms hold:

For any a, b, c in B:

1. closure:	$a + b$ is in B	$a \cdot b$ is in B
2. commutativity:	$a + b = b + a$	$a \cdot b = b \cdot a$
3. associativity:	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
4. distributivity:	$a + (b \cdot c) = (a + b) \cdot (a + c)$	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
5. identity:	$a + 0 = a$	$a \cdot 1 = a$
6. complementarity:	$a + a' = 1$	$a \cdot a' = 0$
7. null:	$a + 1 = 1$	$a \cdot 0 = 0$
8. idempotency:	$a + a = a$	$a \cdot a = a$
9. involution:	$(a')' = a$	

A Combinational Logic Example

Sessions of Class:

We would like to compute the number of lectures or quiz sections remaining *at the start* of a given day of the week.

- **Inputs:** Day of the Week, Lecture/Section flag
- **Output:** Number of sessions left

Examples: Input: (Wednesday, Lecture) Output: 2

Input: (Monday, Section) Output: 1

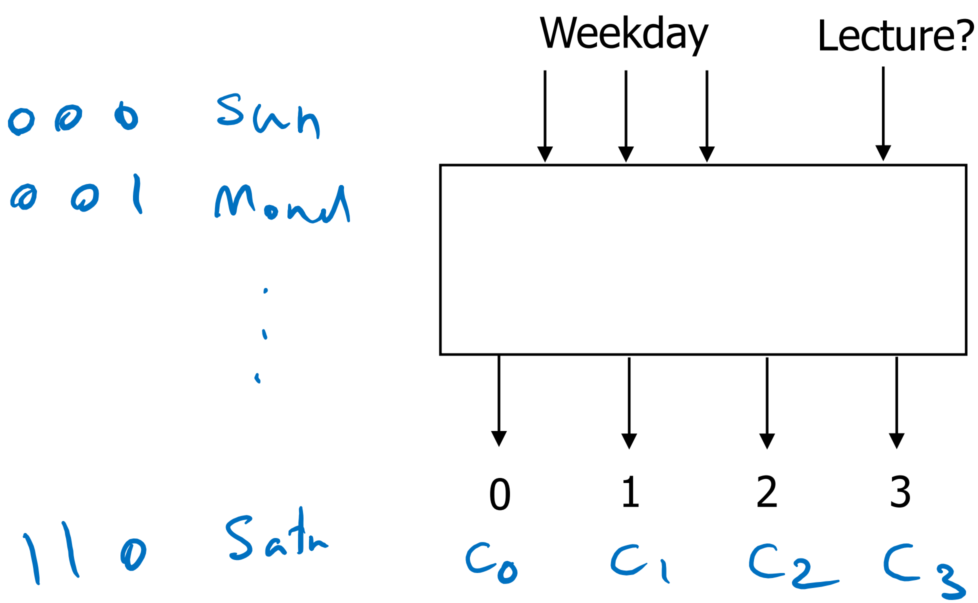
Implementation in Software

```
public int classesLeftInMorning(weekday, lecture_flag) {
    switch (weekday) {
        case SUNDAY:
        case MONDAY:
            return lecture_flag ? 3 : 1;
        case TUESDAY:
        case WEDNESDAY:
            return lecture_flag ? 2 : 1;
        case THURSDAY:
            return lecture_flag ? 1 : 1;
        case FRIDAY:
            return lecture_flag ? 1 : 0;
        case SATURDAY:
            return lecture_flag ? 0 : 0;
  }
}
```

Implementation with Combinational Logic

Encoding:

- How many bits for each input/output?
- Binary number for weekday
- One bit for each possible output



Defining Our Inputs!

Weekday Input:

- Binary number for weekday
- Sunday = 0, Monday = 1, …
- We care about these in binary:

Weekday	Number	Binary
Sunday	0	$(000)_2$
Monday	1	$(001)_2$
Tuesday	2	$(010)_2$
Wednesday	3	$(011)_2$
Thursday	4	$(100)_2$
Friday	5	$(101)_2$
Saturday	6	$(110)_2$

Converting to a Truth Table!

```
case SUNDAY or MONDAY:
   return lecture_flag ? 3 : 1;
case TUESDAY or WEDNESDAY:
   return lecture_flag ? 2 : 1;
case THURSDAY:
   return lecture_flag ? 1 : 1;
case FRIDAY:
   return lecture_flag ? 1 : 0;
case SATURDAY:
   return lecture_flag ? 0 : 0;
```

Weekday		Lecture?	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0				
MON	001	1				
TUE	010	0				
TUE	010	1				
WED	011	0				
WED	011	1				
THU	100	-				
FRI	101	0				
FRI	101	1				
SAT	110	-				
-	111	-				

Converting to a Truth Table!

```
case SUNDAY or MONDAY:
    return lecture_flag ? 3 : 1;
case TUESDAY or WEDNESDAY:
    return lecture_flag ? 2 : 1;
case THURSDAY:
    return lecture_flag ? 1 : 1;
case FRIDAY:
    return lecture_flag ? 1 : 0;
case SATURDAY:
    return lecture_flag ? 0 : 0;
```

Weekday		Lecture?	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

Let's begin by finding an expression for $\mathbf{c_3}$. To do this, we look at the rows where $\mathbf{c_3}$ = **1** (true).

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

DAY == SUN && L == 1

DAY == MON && L == 1

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$d_2d_1d_0$ == 000 && L == 1

$d_2d_1d_0$ == 001 && L == 1

Substituting DAY for the binary representation.

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

d_2 == 0 && d_1 == 0 && d_0 == 0 && L == 1

d_2 == 0 && d_1 == 0 && d_0 == 1 && L == 1

Splitting up the bits of the day; so, we can write a formula.

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$d_2 = 0$ $\quad \neg d_2 \wedge \neg d_1 \wedge \neg d_0 \wedge L$

$d_2' \cdot d_1' \cdot d_0' \cdot L$

+

$d_2' \cdot d_1' \cdot d_0 \cdot L$

Replacing with Boolean Algebra…

Truth Table to Logic (Part 1)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$d_2' \cdot d_1' \cdot d_0' \cdot L$

$d_2' \cdot d_1' \cdot d_0 \cdot L$

Either situation causes c_3 to be true. So, we “or” them.

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

Truth Table to Logic (Part 2)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

Now, we do c_2.

$d_2 = 0 \quad d_1 = 1 \quad d_0 = 0, \quad L = 1$

$$d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L = c_2$$

$$d_2' \cdot d_1 \cdot L = c_2$$

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3	
SUN	000	0	0	1	0	0	→ $d_2' \cdot d_1' \cdot d_0' \cdot L'$
SUN	000	1	0	0	0	1	
MON	001	0	0	1	0	0	→ $d_2' \cdot d_1' \cdot d_0 \cdot L'$
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	→ $d_2' \cdot d_1 \cdot d_0' \cdot L'$
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	→ $d_2' \cdot d_1 \cdot d_0 \cdot L'$
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	→ ??? $d_2 \cdot d_1' \cdot d_0'$
FRI	101	0	1	0	0	0	
FRI	101	1	0	1	0	0	→ $d_2 \cdot d_1' \cdot d_0 \cdot L$
SAT	110	-	1	0	0	0	
-	111	-	1	0	0	0	

Now, we do c_1:

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

Now, we do **c_1**:

$d_2' \cdot d_1' \cdot d_0' \cdot L'$

$d_2' \cdot d_1' \cdot d_0 \cdot L'$

$d_2' \cdot d_1 \cdot d_0' \cdot L'$

$d_2' \cdot d_1 \cdot d_0 \cdot L'$

$d_2 \cdot d_1' \cdot d_0'$

$d_2 \cdot d_1' \cdot d_0 \cdot L$

No matter what L is, we always say it's 1. So, we don't need L in the expression.

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

Truth Table to Logic (Part 3)

Now, we do c_1:

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3	
SUN	000	0	0	1	0	0	→ $d_2' \cdot d_1' \cdot d_0' \cdot L'$
SUN	000	1	0	0	0	1	
MON	001	0	0	1	0	0	→ $d_2' \cdot d_1' \cdot d_0 \cdot L'$
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	→ $d_2' \cdot d_1 \cdot d_0' \cdot L'$
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	→ $d_2' \cdot d_1 \cdot d_0 \cdot L'$
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	→ $d_2 \cdot d_1' \cdot d_0'$
FRI	101	0	1	0	0	0	
FRI	101	1	0	1	0	0	→ $d_2 \cdot d_1' \cdot d_0 \cdot L$
SAT	110	-	1	0	0	0	
-	111	-	1	0	0	0	

No matter what L is, we always say it's 1. So, we don't need L in the expression.

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

$$c_1 = d_2' \cdot d_1' \cdot d_0' \cdot L' + d_2' \cdot d_1' \cdot d_0 \cdot L' + d_2' \cdot d_1 \cdot d_0' \cdot L' + d_2' \cdot d_1 \cdot d_0 \cdot L' + d_2 \cdot d_1' \cdot d_0' + d_2 \cdot d_1' \cdot d_0 \cdot L$$

Truth Table to Logic (Part 4)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$$c_1 = d_2' \bullet d_1' \bullet d_0' \bullet L' + d_2' \bullet d_1' \bullet d_0 \bullet L' + d_2' \bullet d_1 \bullet d_0' \bullet L' + d_2' \bullet d_1 \bullet d_0 \bullet L' + d_2 \bullet d_1' \bullet d_0' + d_2 \bullet d_1' \bullet d_0 \bullet L$$

$$c_2 = d_2' \bullet d_1 \bullet d_0' \bullet L + d_2' \bullet d_1 \bullet d_0 \bullet L$$

$$c_3 = d_2' \bullet d_1' \bullet d_0' \bullet L + d_2' \bullet d_1' \bullet d_0 \bullet L$$

Finally, we do c_0:

FRI 101 0 → $d_2 \bullet d_1' \bullet d_0 \bullet L'$

SAT 110 → $d_2 \bullet d_1 \bullet d_0'$

111 → $d_2 \bullet d_1 \bullet d_0$

Truth Table to Logic (Part 4)

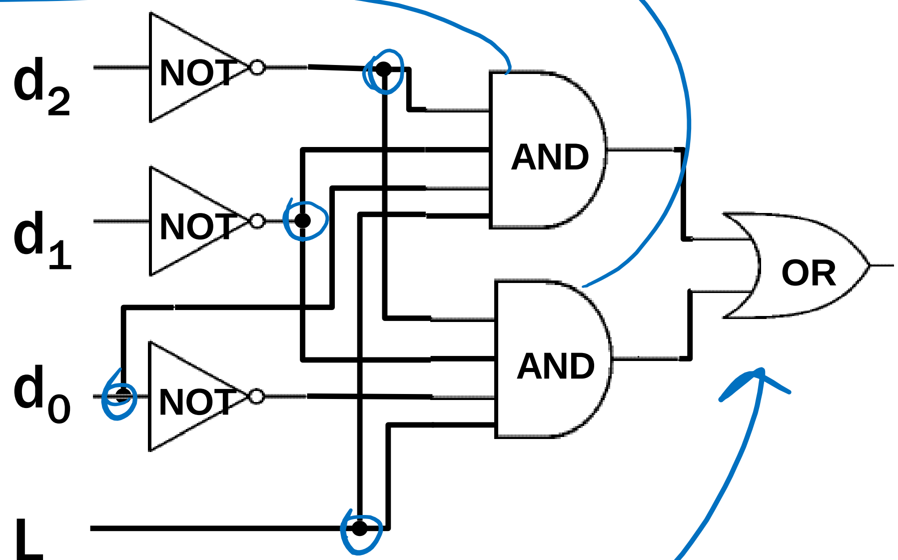
$c_0 = d_2 \cdot d_1' \cdot d_0 \cdot L' + d_2 \cdot d_1 \cdot d_0' + d_2 \cdot d_1 \cdot d_0$

$c_1 = d_2' \cdot d_1' \cdot d_0' \cdot L' + d_2' \cdot d_1' \cdot d_0 \cdot L' + d_2' \cdot d_1 \cdot d_0' \cdot L' + d_2' \cdot d_1 \cdot d_0 \cdot L' + d_2 \cdot d_1' \cdot d_0' + d_2 \cdot d_1' \cdot d_0 \cdot L$

$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$

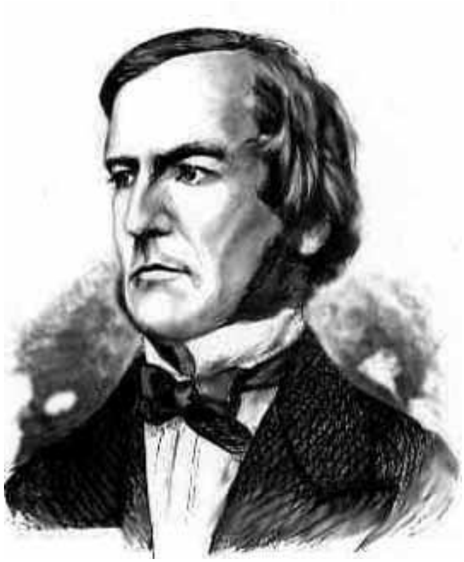
$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$

Here's c_3 as a circuit:



Boolean Algebra

- **Usual notation used in circuit design**
- **Boolean algebra**
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { ' }
  - such that the following axioms hold:



For any a, b, c in B:

1. closure:	a + b is in B	a • b is in B
2. commutativity:	a + b = b + a	a • b = b • a
3. associativity:	a + (b + c) = (a + b) + c	a • (b • c) = (a • b) • c
4. distributivity:	a + (b • c) = (a + b) • (a + c)	a • (b + c) = (a • b) + (a • c)
5. identity:	a + 0 = a	a • 1 = a
6. complementarity:	a + a' = 1	a • a' = 0
7. null:	a + 1 = 1	a • 0 = 0
8. idempotency:	a + a = a	a • a = a
9. involution:	(a')' = a	

Simplification using Boolean Algebra

uniting:

10. $a \cdot b + a \cdot b' = a$

10D. $(a + b) \cdot (a + b') = a$

absorption:

11. $a + a \cdot b = a$

11D. $a \cdot (a + b) = a$

12. $(a + b') \cdot b = a \cdot b$

12D. $(a \cdot b') + b = a + b$

factoring:

13. $(a + b) \cdot (a' + c) = a \cdot c + a' \cdot b$

13D. $a \cdot b + a' \cdot c = (a + c) \cdot (a' + b)$

consensus:

14. $(a \cdot b) + (b \cdot c) + (a' \cdot c) = a \cdot b + a' \cdot c$

14D. $(a + b) \cdot (b + c) \cdot (a' + c) = (a + b) \cdot (a' + c)$

de Morgan's:

15. $(a + b + \ldots)' = a' \cdot b' \cdot \ldots$

15D. $(a \cdot b \cdot \ldots)' = a' + b' + \ldots$

Proving Theorems

2. commutativity: $a + b = b + a$ $a \cdot b = b \cdot a$
3. associativity: $a + (b + c) = (a + b) + c$ $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
4. distributivity: $a + (b \cdot c) = (a + b) \cdot (a + c)$ $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
5. identity: $a + 0 = a$ $a \cdot 1 = a$
6. complementarity: $a + a' = 1$ $a \cdot a' = 0$
7. null: $a + 1 = 1$ $a \cdot 0 = 0$
8. idempotency: $a + a = a$ $a \cdot a = a$
9. involution: $(a')' = a$

Using the laws of Boolean Algebra:

prove the Uniting theorem: $X \cdot Y + X \cdot Y' = X$

$X \cdot Y + X \cdot Y' = X \cdot (Y + Y')$ distrib

$= X \cdot 1$ complem

$= X$ identity

prove the Absorption theorem: $(X) + (X \cdot Y) = X$

$X + X \cdot Y = X \cdot 1 + X \cdot Y$ identity

$= X \cdot (1 + Y)$ distri

$= X \cdot 1$ Null

$= X$ identity

Proving Theorems

2. commutativity:	$a + b = b + a$	$a \cdot b = b \cdot a$
3. associativity:	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
4. distributivity:	$a + (b \cdot c) = (a + b) \cdot (a + c)$	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
5. identity:	$a + 0 = a$	$a \cdot 1 = a$
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8. idempotency:	$a + a = a$	$a \cdot a = a$
9. involution:	$(a')' = a$	

Using the laws of Boolean Algebra:

prove the Uniting theorem: $X \cdot Y + X \cdot Y' = X$

distributivity	$X \cdot Y + X \cdot Y' = X \cdot (Y + Y')$
complementarity	$= X \cdot 1$
identity	$= X$

prove the theorem: $X + X \cdot Y = X$

identity	$X + X \cdot Y = X \cdot 1 + X \cdot Y$
distributivity	$= X \cdot (1 + Y)$
commutativity	$= X \cdot (Y + 1)$
null	$= X \cdot 1$
identity	$= X$

Proving Theorems

Using complete truth table:

For example, de Morgan’s Law:

$(X + Y)' = X' \bullet Y'$

NOR is equivalent to AND with inputs complemented

X	Y	X′	Y′	(X + Y)′	X′ • Y′
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

$(X \bullet Y)' = X' + Y'$

NAND is equivalent to OR with inputs complemented

X	Y	X′	Y′	(X • Y)′	X′ + Y′
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

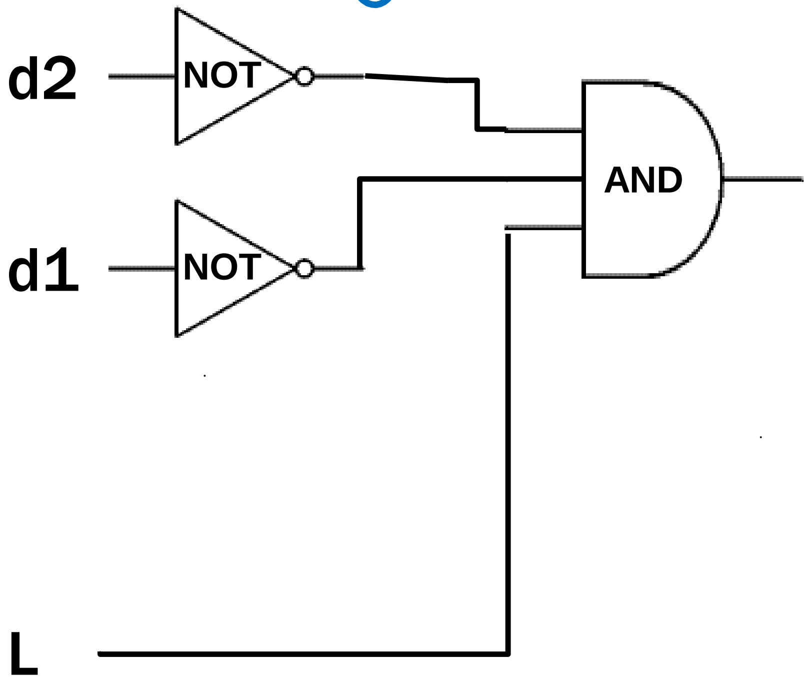
Simplifying using Boolean Algebra

$c3 = d2' \bullet d1' \bullet d0' \bullet L \ + \ d2' \bullet d1' \bullet d0 \bullet L$

$\quad = d2' \bullet d1' \bullet (d0' \ + \ d0) \bullet L$ dist

$\quad = d2' \bullet d1' \bullet 1 \bullet L$ compl

$\quad = d2' \bullet d1' \bullet L$ identy



1-bit Binary Adder

$$\begin{array}{r} A \\ + \underline{B} \\ S \\ (C_{OUT}) \end{array}$$

$0 + 0 = 0$ (with $C_{OUT} = 0$)

$0 + 1 = 1$ (with $C_{OUT} = 0$)

$1 + 0 = 1$ (with $C_{OUT} = 0$)

$1 + 1 = 0$ (with $C_{OUT} = 1$)

S

Cout

1-bit Binary Adder

A	$0 + 0 = 0$ (with $C_{OUT} = 0$)
+ B	$0 + 1 = 1$ (with $C_{OUT} = 0$)
S	$1 + 0 = 1$ (with $C_{OUT} = 0$)
(C_{OUT})	$1 + 1 = 0$ (with $C_{OUT} = 1$)

Idea: To chain these together, let's add a carry-in

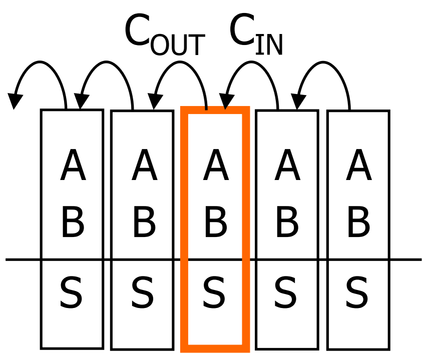
1-bit Binary Adder

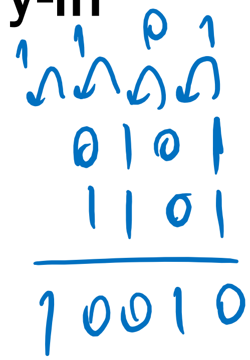
A
+ B
S
(C_{OUT})

0 + 0 = 0 (with C_{OUT} = 0)
0 + 1 = 1 (with C_{OUT} = 0)
1 + 0 = 1 (with C_{OUT} = 0)
1 + 1 = 0 (with C_{OUT} = 1)

Idea: To chain these together, let's add a carry-in

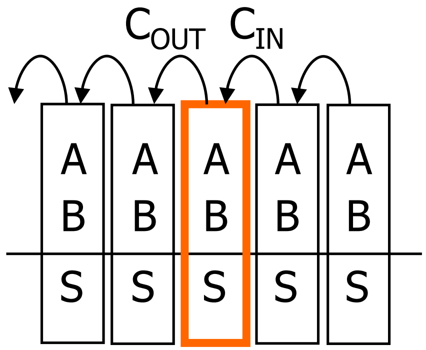
(C_{IN})
A
+ B
S
(C_{OUT})





1-bit Binary Adder

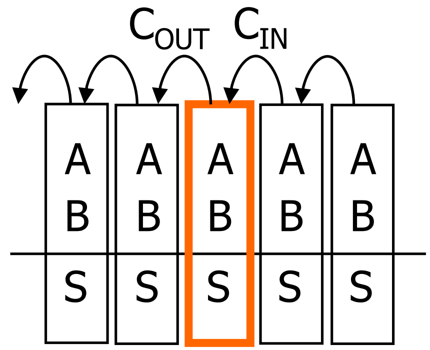
- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Derive an expression for S

$A' \cdot B' \cdot C_{IN}$

$A' \cdot B \cdot C_{IN}'$

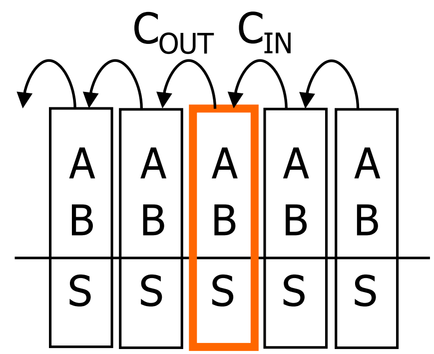
$A \cdot B' \cdot C_{IN}'$

$A \cdot B \cdot C_{IN}$

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Derive an expression for C_{OUT}

$A' \cdot B \cdot C_{IN}$

$A \cdot B' \cdot C_{IN}$

$A \cdot B \cdot C_{IN}'$

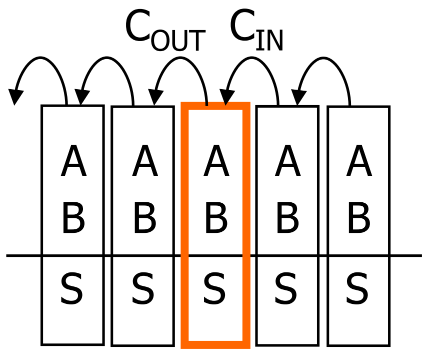
$A \cdot B \cdot C_{IN}$

$$C_{OUT} = A' \cdot B \cdot C_{IN} + A \cdot B' \cdot C_{IN} + A \cdot B \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

1-bit Binary Adder

- **Inputs:** A, B, Carry-in
- **Outputs:** Sum, Carry-out



A	B	C_{IN}	C_{OUT}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

$$S = A' \cdot B' \cdot C_{IN} + A' \cdot B \cdot C_{IN}' + A \cdot B' \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

$$C_{OUT} = A' \cdot B \cdot C_{IN} + A \cdot B' \cdot C_{IN} + A \cdot B \cdot C_{IN}' + A \cdot B \cdot C_{IN}$$

Apply Theorems to Simplify Expressions

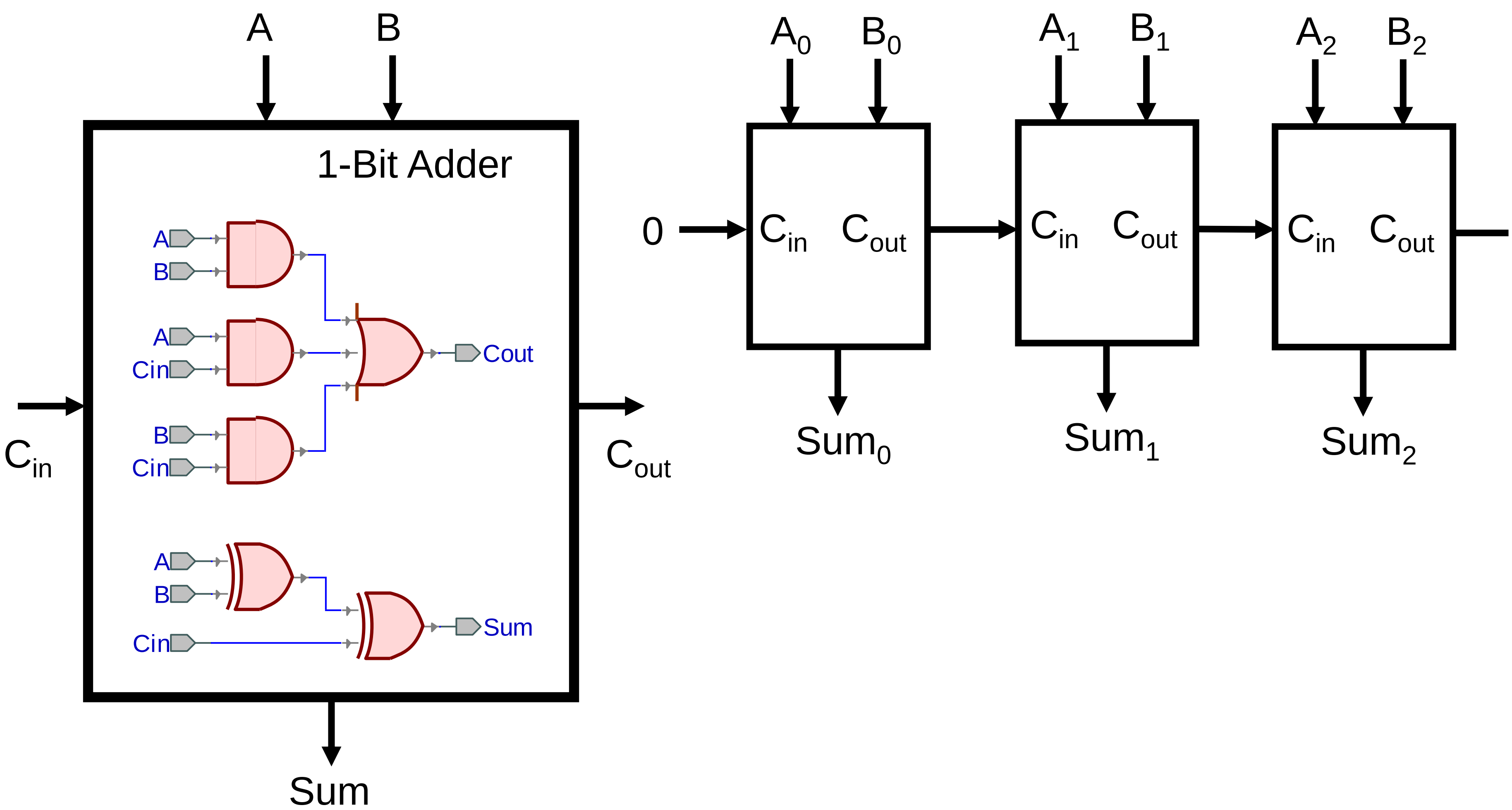
The theorems of Boolean algebra can simplify expressions

– **e.g., full adder's carry-out function**

Cout = A′ B Cin + A B′ Cin + A B Cin′ + A B Cin
= A′ B Cin + A B′ Cin + A B Cin′ + A B Cin + A B Cin
= A′ B Cin + A B Cin + A B′ Cin + A B Cin′ + A B Cin
= (A′ + A) B Cin + A B′ Cin + A B Cin′ + A B Cin
= (1) B Cin + A B′ Cin + A B Cin′ + A B Cin
= B Cin + A B′ Cin + A B Cin′ + A B Cin + A B Cin
= B Cin + A B′ Cin + A B Cin + A B Cin′ + A B Cin
= B Cin + A (B′ + B) Cin + A B Cin′ + A B Cin
= B Cin + A (1) Cin + A B Cin′ + A B Cin
= B Cin + A Cin + A B (Cin′ + Cin)
= B Cin + A Cin + A B (1)
= B Cin + A Cin + A B

adding extra terms creates new factoring opportunities

A 2-bit Ripple-Carry Adder



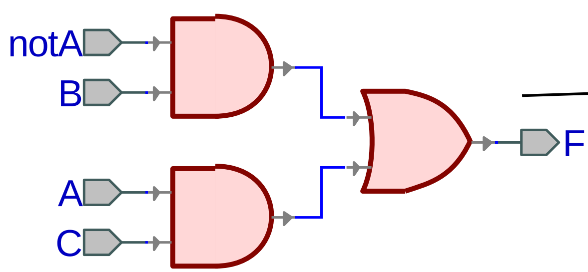
Mapping Truth Tables to Logic Gates

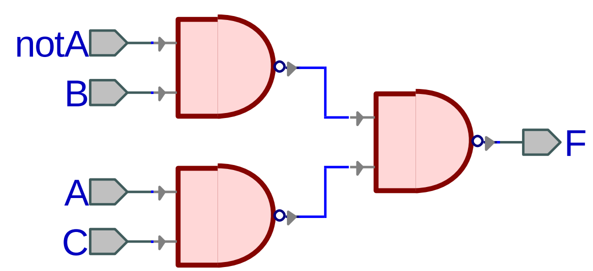
Given a truth table:

1. **Write the Boolean expression**
2. **Minimize the Boolean expression**
3. **Draw as gates**
4. **Map to available gates**

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$F = A'BC' + A'BC + AB'C + ABC$$
$$= A'B(C' + C) + AC(B' + B)$$
$$= A'B + AC$$





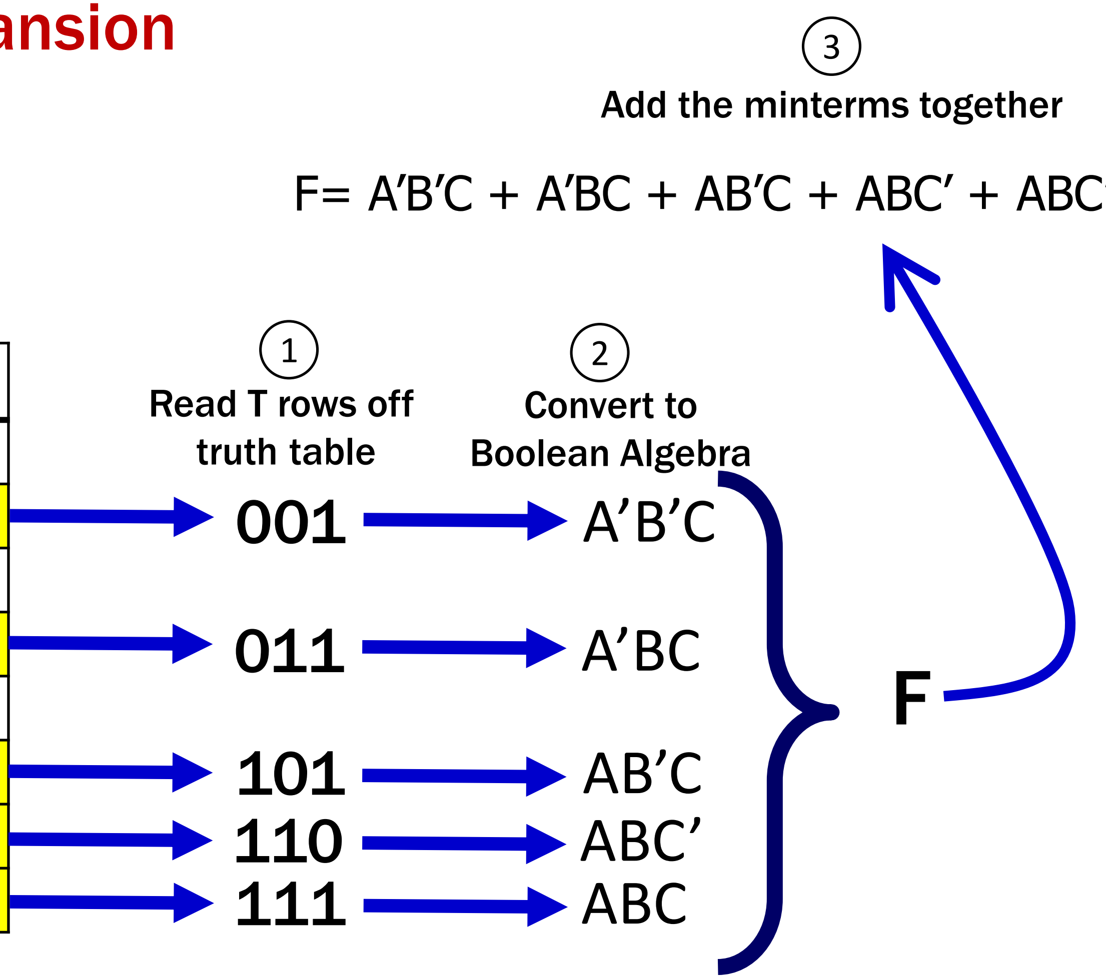
Canonical Forms

- Truth table is the **unique signature** of a Boolean Function
- The same truth table can have many gate realizations
  - We've seen this already
  - Depends on how good we are at Boolean simplification
- Canonical forms
  - Standard forms for a Boolean expression
  - We all come up with the same expression

Sum-of-Products Canonical Form

- AKA **Disjunctive Normal Form (DNF)**
- AKA **Minterm Expansion**

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1



Sum-of-Products Canonical Form

Product term (or minterm)

- **ANDed product of literals – input combination for which output is true**
- **each variable appears exactly once, true or inverted (but not both)**

A	B	C	minterms
0	0	0	A′B′C′
0	0	1	A′B′C
0	1	0	A′BC′
0	1	1	A′BC
1	0	0	AB′C′
1	0	1	AB′C
1	1	0	ABC′
1	1	1	ABC

F in canonical form:

F(A, B, C) = A′B′C + A′BC + AB′C + ABC′ + ABC

canonical form ≠ minimal form

F(A, B, C) = A′B′C + A′BC + AB′C + ABC + ABC′
= (A′B′ + A′B + AB′ + AB)C + ABC′
= ((A′ + A)(B′ + B))C + ABC′
= C + ABC′
= ABC′ + C
= AB + C

Product-of-Sums Canonical Form

- AKA Conjunctive Normal Form (CNF)
- AKA Maxterm Expansion

④ Multiply the maxterms together

F =

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

① Read F rows off truth table

② Negate all bits

③ Convert to Boolean Algebra

F

Product-of-Sums Canonical Form

- AKA **Conjunctive Normal Form (CNF)**
- AKA **Maxterm Expansion**

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

① Read F rows off truth table → ② Negate all bits → ③ Convert to Boolean Algebra

000 → 111 → A + B + C

010 → 101 → A + B′ + C

100 → 011 → A′ + B + C

④ Multiply the maxterms together

F = (A + B + C)(A + B′ + C)(A′ + B + C)

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F’)’ = F
- We know how to get a **minterm** expansion for F’

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F')' = F
- We know how to get a **minterm** expansion for F'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Taking the complement of both sides…

$$(F')' = (A'B'C' + A'BC' + AB'C')'$$

And using DeMorgan/Comp….

$$F = (A'B'C')' \ (A'BC')' \ (AB'C')'$$

$$F = (A + B + C)(A + B' + C)(A' + B + C)$$

Product-of-Sums Canonical Form

Sum term (or maxterm)

- ORed sum of literals – input combination for which output is false
- each variable appears exactly once, true or inverted (but not both)

A	B	C	maxterms
0	0	0	A+B+C
0	0	1	A+B+C′
0	1	0	A+B′+C
0	1	1	A+B′+C′
1	0	0	A′+B+C
1	0	1	A′+B+C′
1	1	0	A′+B′+C
1	1	1	A′+B′+C′

F in canonical form:

F(A, B, C) = (A + B + C) (A + B′ + C) (A′ + B + C)

canonical form ≠ minimal form

F(A, B, C) = (A + B + C) (A + B′ + C) (A′ + B + C)
= (A + B + C) (A + B′ + C)
(A + B + C) (A′ + B + C)
= (A + C) (B + C)