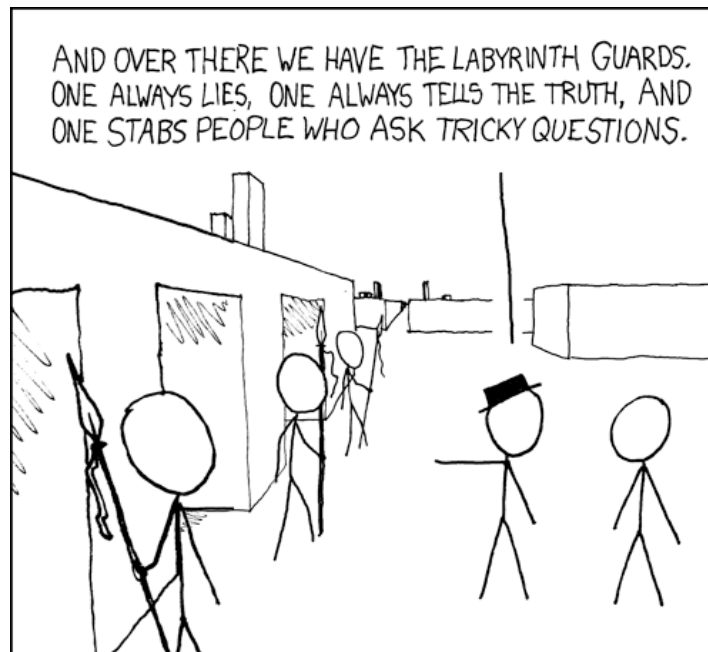


# CSE 311: Foundations of Computing

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## Lecture 3: Digital Circuits & Equivalence



# Homework #1

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- You should have received
  - An e-mail from [cse311] with information pointing you to Canvas to submit HW
  - An e-mail from UW Canvas with a notification about the homework assignment. Click on “Assignments” to see all the questions

**If you haven't received one, send e-mail to [cse311-staff@cs.washington.edu](mailto:cse311-staff@cs.washington.edu)**

## Last class: Logical Equivalence $A \equiv B$

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$A \equiv B$  is an assertion that *two propositions*  $A$  and  $B$  always have the same truth values.

$A \equiv B$  and  $(A \leftrightarrow B) \equiv T$  have the same meaning.

tautology

$$p \wedge q \equiv q \wedge p$$

$p$	$q$	$p \wedge q$	$q \wedge p$	$(p \wedge q) \leftrightarrow (q \wedge p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

$$p \wedge q \not\equiv q \vee p$$

When  $p=T$  and  $q=F$ ,  $p \wedge q$  is false, but  $q \vee p$  is true

# Last class: De Morgan's Laws

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## De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$



# Last class: Equivalences Related to Implication

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## Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

## Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

## Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

# Last class: Properties of Logical Connectives

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- **Identity**

- $p \wedge T \equiv p$

- $p \vee F \equiv p$

- **Domination**

- $p \vee T \equiv T$

- $p \wedge F \equiv F$

- **Idempotent**

- $p \vee p \equiv p$

- $p \wedge p \equiv p$

- **Commutative**

- $p \vee q \equiv q \vee p$

- $p \wedge q \equiv q \wedge p$

- **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$

- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- **Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **Absorption**

- $p \vee (p \wedge q) \equiv p$

- $p \wedge (p \vee q) \equiv p$

- **Negation**

- $p \vee \neg p \equiv T$

- $p \wedge \neg p \equiv F$

# One more easy equivalence

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## Double Negation

$$p \equiv \neg \neg p$$

$p$	$\neg p$	$\neg \neg p$	$p \leftrightarrow \neg \neg p$
T	F	T	T
F	T	F	T

# Last class: Digital Circuits

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## Computing With Logic

- **T** corresponds to **1** or “high” voltage
- **F** corresponds to **0** or “low” voltage

## Gates

- Take inputs and produce outputs (functions)
- Several kinds of gates
- Correspond to propositional connectives (most of them)



# Last class: AND, OR, NOT Gates

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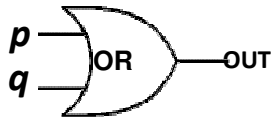
## AND Gate



$p$	$q$	OUT
1	1	1
1	0	0
0	1	0
0	0	0

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

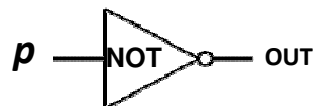
## OR Gate



$p$	$q$	OUT
1	1	1
1	0	1
0	1	1
0	0	0

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

## NOT Gate

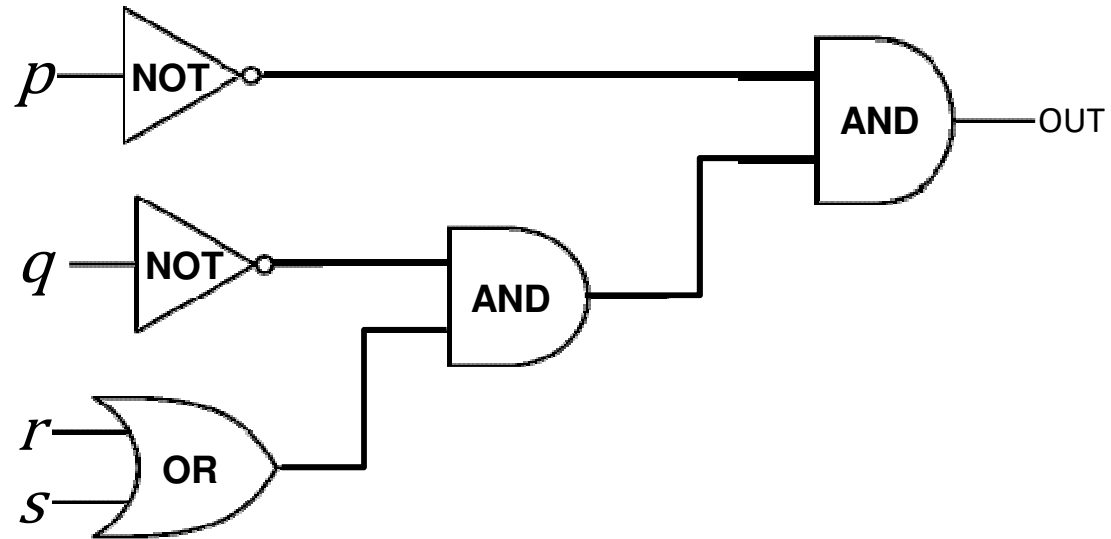


$p$	OUT
1	0
0	1

$p$	$\neg p$
T	F
F	T

# Combinational Logic Circuits

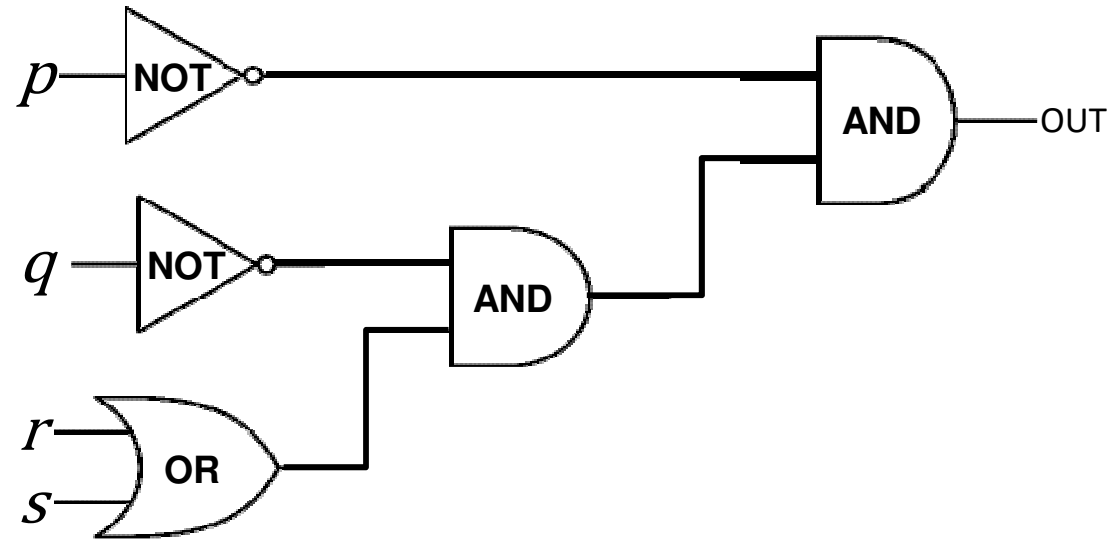
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**Values get sent along wires connecting gates**

# Combinational Logic Circuits

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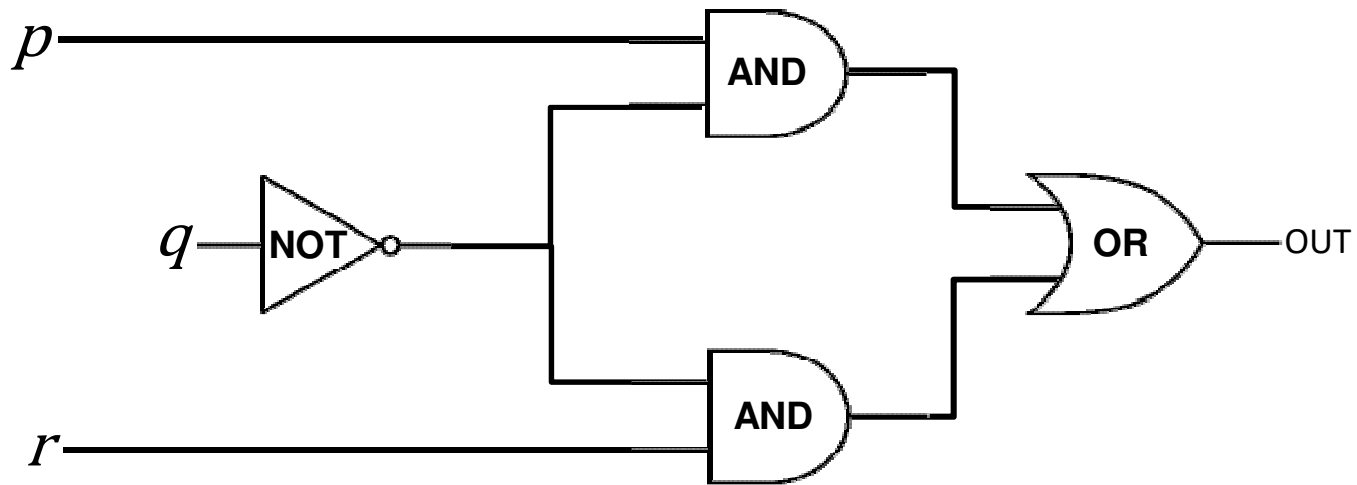


**Values get sent along wires connecting gates**

$$\neg p \wedge (\neg q \wedge (r \vee s))$$

# Combinational Logic Circuits

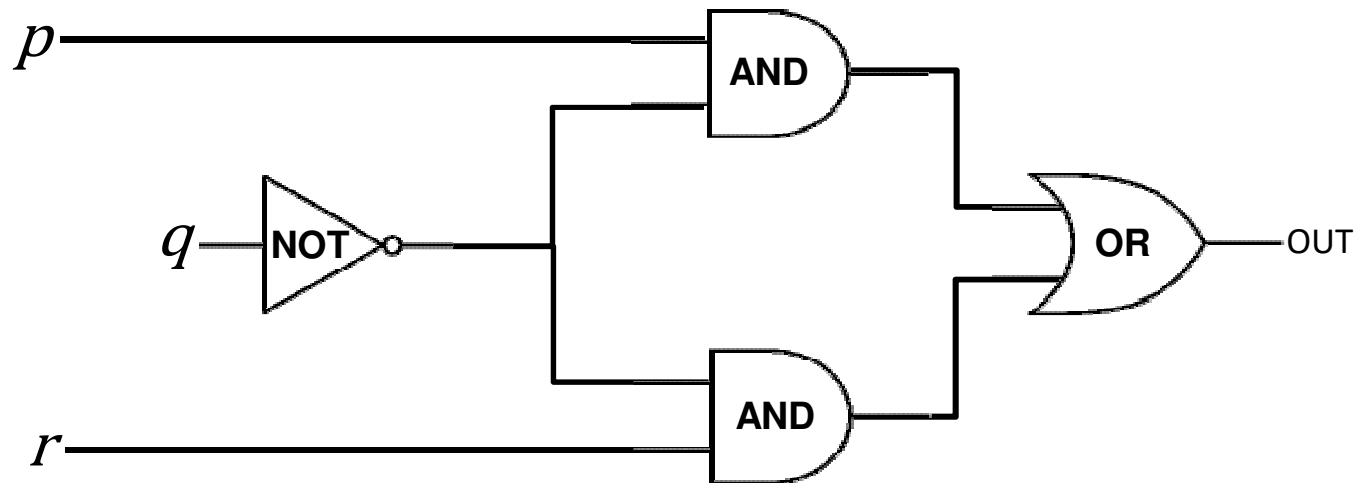
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**Wires can send one value to multiple gates!**

# Combinational Logic Circuits

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**Wires can send one value to multiple gates!**

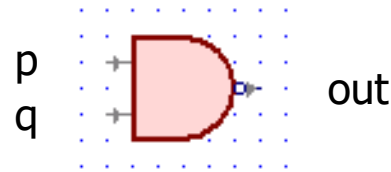
$$(p \wedge \neg q) \vee (\neg q \wedge r)$$

# Other Useful Gates

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## NAND

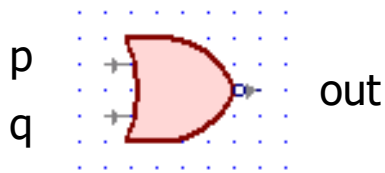
$$\neg(p \wedge q)$$



p	q	out
0	0	1
0	1	1
1	0	1
1	1	0

## NOR

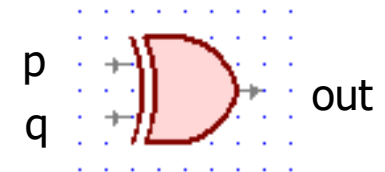
$$\neg(p \vee q)$$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	0

## XOR

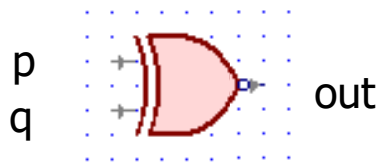
$$p \oplus q$$



p	q	out
0	0	0
0	1	1
1	0	1
1	1	0

## XNOR

$$p \leftrightarrow q$$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	1

# Understanding logic and circuits

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**When do two logic formulas mean the same thing?**

**When do two circuits compute the same function?**

**What logical properties can we infer from other ones?**

# Basic rules of reasoning and logic

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- **Allow manipulation of logical formulas**
  - Simplification
  - Testing for equivalence
- **Applications**
  - Query optimization
  - Search optimization and caching
  - Artificial Intelligence
  - Program verification



# Computing Equivalence

---

Given two propositions, can we write an algorithm to determine if they are equivalent?

What is the runtime of our algorithm?

# Computing Equivalence

---

Given two propositions, can we write an algorithm to determine if they are equivalent?

Yes! Generate the truth tables for both propositions and check if they are the same for every entry.

What is the runtime of our algorithm?

Every atomic proposition has two possibilities (T, F). If there are  $n$  atomic propositions, there are  $2^n$  rows in the truth table.

## Another approach: Logical Proofs

---

### To show A is equivalent to B

- Apply a series of logical equivalences to sub-expressions to convert A to B

### To show A is a tautology

- Apply a series of logical equivalences to sub-expressions to convert A to T

# Another approach: Logical Proofs

---

## To show A is equivalent to B

- Apply a series of logical equivalences to sub-expressions to convert A to B

Example:

Let A be “ $(p \wedge p) \vee p$ ”, and B be “ $p$ ”.

Our general proof looks like:

$$\begin{aligned} (p \wedge p) \vee p &\equiv ( && ) \\ &\equiv p \end{aligned}$$

# Another approach: Logical Proofs

---

- **Identity**

- $p \wedge T \equiv p$

- $p \vee F \equiv p$

- **Domination**

- $p \vee T \equiv T$

- $p \wedge F \equiv F$

- **Idempotent**

- $p \vee p \equiv p$

- $p \wedge p \equiv p$

- **Commutative**

- $p \vee q \equiv q \vee p$

- $p \wedge q \equiv q \wedge p$

- **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$

- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- **Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **Absorption**

- $p \vee (p \wedge q) \equiv p$

- $p \wedge (p \vee q) \equiv p$

- **Negation**

- $p \vee \neg p \equiv T$

- $p \wedge \neg p \equiv F$

- **De Morgan's Laws**

- $\neg(p \wedge q) \equiv \neg p \vee \neg q$

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$

- **Law of Implication**

- $p \rightarrow q \equiv \neg p \vee q$

- **Contrapositive**

- $p \rightarrow q \equiv \neg q \rightarrow \neg p$

- **Biconditional**

- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

- **Double Negation**

- $p \equiv \neg \neg p$

## Example:

Let A be “ $(p \wedge p) \vee p$ ”, and B be “ $p$ ”.

Our general proof looks like:

$$\begin{aligned} (p \wedge p) \vee p &\equiv ( && ) \\ &\equiv p \end{aligned}$$

# Logical Proofs

---

- **Identity**

- $p \wedge T \equiv p$
- $p \vee F \equiv p$

- **Domination**

- $p \vee T \equiv T$
- $p \wedge F \equiv F$

- **Idempotent**

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

- **Commutative**

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

- **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- **Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **Absorption**

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

- **Negation**

- $p \vee \neg p \equiv T$
- $p \wedge \neg p \equiv F$

- **De Morgan's Laws**

$$\begin{aligned}\neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q\end{aligned}$$

- **Law of Implication**

$$p \rightarrow q \equiv \neg p \vee q$$

- **Contrapositive**

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

- **Biconditional**

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

- **Double Negation**

$$p \equiv \neg \neg p$$

## Example:

Let A be “ $(p \wedge p) \vee p$ ”, and B be “ $p$ ”.

Our general proof looks like:

$$\begin{aligned}(p \wedge p) \vee p &\equiv (p \vee p) && \text{Idempotent} \\ &\equiv p && \text{Idempotent}\end{aligned}$$

# Logical Proofs

---

## To show A is a tautology

- Apply a series of logical equivalences to sub-expressions to convert A to T

Example:

Let A be “ $\neg p \vee (p \vee p)$ ”.

Our general proof looks like:

$$\begin{aligned}\neg p \vee (p \vee p) &\equiv ( && ) \\ &\equiv \\ &\equiv \mathbf{T}\end{aligned}$$

# Logical Proofs

---

- **Identity**

- $p \wedge T \equiv p$
- $p \vee F \equiv p$

- **Domination**

- $p \vee T \equiv T$
- $p \wedge F \equiv F$

- **Idempotent**

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

- **Commutative**

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

- **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- **Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **Absorption**

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

- **Negation**

- $p \vee \neg p \equiv T$
- $p \wedge \neg p \equiv F$

- **De Morgan's Laws**

$$\begin{aligned}\neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q\end{aligned}$$

- **Law of Implication**

$$p \rightarrow q \equiv \neg p \vee q$$

- **Contrapositive**

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

- **Biconditional**

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

- **Double Negation**

$$p \equiv \neg \neg p$$

## Example:

Let A be “ $\neg p \vee (p \vee p)$ ”.

Our general proof looks like:

$$\begin{aligned}\neg p \vee (p \vee p) &\equiv ( && ) \\ &\equiv && \\ &\equiv \mathbf{T}\end{aligned}$$



# Logical Proofs

---

- **Identity**

- $p \wedge \text{T} \equiv p$
- $p \vee \text{F} \equiv p$

- **Domination**

- $p \vee \text{T} \equiv \text{T}$
- $p \wedge \text{F} \equiv \text{F}$

- **Idempotent**

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

- **Commutative**

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

- **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- **Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **Absorption**

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

- **Negation**

- $p \vee \neg p \equiv \text{T}$
- $p \wedge \neg p \equiv \text{F}$

## De Morgan's Laws

$$\begin{aligned}\neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q\end{aligned}$$

## Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

## Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

## Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

## Double Negation

$$p \equiv \neg \neg p$$

## Example:

Let A be “ $\neg p \vee (p \vee p)$ ”.

Our general proof looks like:

$$\begin{aligned}\neg p \vee (p \vee p) &\equiv (\neg p \vee p) && \text{Idempotent} \\ &\equiv p \vee \neg p && \text{Commutative} \\ &\equiv \text{T} && \text{Negation}\end{aligned}$$

# Prove these propositions are equivalent: Option 1

---

**Prove:**  $p \wedge (p \rightarrow q) \equiv p \wedge q$

**Make a Truth Table and show:**

$$(p \wedge (p \rightarrow q)) \leftrightarrow (p \wedge q) \equiv \mathbf{T}$$

$p$	$q$	$p \rightarrow q$	$(p \wedge (p \rightarrow q))$	$p \wedge q$	$(p \wedge (p \rightarrow q)) \leftrightarrow (p \wedge q)$
T	T	T	T	T	T
T	F	F	F	F	T
F	T	T	F	F	T
F	F	T	F	F	T

# Prove these propositions are equivalent: Option 2

---

Prove:  $p \wedge (p \rightarrow q) \equiv p \wedge q$

$$\begin{aligned} p \wedge (p \rightarrow q) &\equiv \\ &\equiv \\ &\equiv \\ &\equiv \\ &\equiv p \wedge q \end{aligned}$$

---

- **Identity**

- $p \wedge T \equiv p$
- $p \vee F \equiv p$

- **Domination**

- $p \vee T \equiv T$
- $p \wedge F \equiv F$

- **Idempotent**

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

- **Commutative**

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

- **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- **Distributive**

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- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **Absorption**

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

- **Negation**

- $p \vee \neg p \equiv T$
- $p \wedge \neg p \equiv F$

- **De Morgan's Laws**

$$\begin{aligned} \neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q \end{aligned}$$

- **Law of Implication**

$$p \rightarrow q \equiv \neg p \vee q$$

- **Contrapositive**

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

- **Biconditional**

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

- **Double Negation**

$$p \equiv \neg \neg p$$

# Prove these propositions are equivalent: Option 2

---

**Prove:  $p \wedge (p \rightarrow q) \equiv p \wedge q$**

$$\begin{aligned} p \wedge (p \rightarrow q) &\equiv p \wedge (\neg p \vee q) \\ &\equiv (p \wedge \neg p) \vee (p \wedge q) \\ &\equiv \mathbf{F} \vee (p \wedge q) \\ &\equiv (p \wedge q) \vee \mathbf{F} \\ &\equiv p \wedge q \end{aligned}$$

Law of Implication

Distributive

Negation

Commutative

Identity

---

• **Identity**

- $p \wedge \mathbf{T} \equiv p$
- $p \vee \mathbf{F} \equiv p$

• **Domination**

- $p \vee \mathbf{T} \equiv \mathbf{T}$
- $p \wedge \mathbf{F} \equiv \mathbf{F}$

• **Idempotent**

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

• **Commutative**

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

• **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

• **Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

• **Absorption**

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

• **Negation**

- $p \vee \neg p \equiv \mathbf{T}$
- $p \wedge \neg p \equiv \mathbf{F}$

**De Morgan's Laws**

$$\begin{aligned} \neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q \end{aligned}$$

**Law of Implication**

$$p \rightarrow q \equiv \neg p \vee q$$

**Contrapositive**

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

**Biconditional**

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

**Double Negation**

$$p \equiv \neg \neg p$$

# Prove this is a Tautology: Option 1

---

$$(p \wedge q) \rightarrow (q \vee p)$$

Make a Truth Table and show:

$$(p \wedge q) \rightarrow (q \vee p) \equiv \mathbf{T}$$

$p$	$q$	$p \wedge q$	$q \vee p$	$(p \wedge q) \rightarrow (q \vee p)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

# Prove this is a Tautology: Option 2

$$(p \wedge q) \rightarrow (q \vee p)$$

Use a series of equivalences like so:

$$\begin{aligned} (p \wedge q) \rightarrow (q \vee p) &\equiv \\ &\equiv \\ &\equiv \\ &\equiv \\ &\equiv \\ &\equiv \\ &\equiv \\ &\equiv \\ &\equiv \\ &\equiv \quad \mathbf{T} \end{aligned}$$

**Identity**

- $p \wedge T \equiv p$
- $p \vee F \equiv p$

**Domination**

- $p \vee T \equiv T$
- $p \wedge F \equiv F$

**Idempotent**

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

**Commutative**

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

**Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

**Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

**Absorption**

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

**Negation**

- $p \vee \neg p \equiv T$
- $p \wedge \neg p \equiv F$

# Prove this is a Tautology: Option 2

$$(p \wedge q) \rightarrow (q \vee p)$$

Use a series of equivalences like so:

$$\begin{aligned}(p \wedge q) \rightarrow (q \vee p) &\equiv \neg(p \wedge q) \vee (q \vee p) \\ &\equiv (\neg p \vee \neg q) \vee (q \vee p) \\ &\equiv \neg p \vee (\neg q \vee (q \vee p)) \\ &\equiv \neg p \vee ((\neg q \vee q) \vee p) \\ &\equiv \neg p \vee (p \vee (\neg q \vee q)) \\ &\equiv (\neg p \vee p) \vee (\neg q \vee q) \\ &\equiv (p \vee \neg p) \vee (q \vee \neg q) \\ &\equiv \mathbf{T} \vee \mathbf{T} \\ &\equiv \mathbf{T}\end{aligned}$$

## Identity

- $p \wedge \mathbf{T} \equiv p$
- $p \vee \mathbf{F} \equiv p$

## Domination

- $p \vee \mathbf{T} \equiv \mathbf{T}$
- $p \wedge \mathbf{F} \equiv \mathbf{F}$

## Idempotent

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

## Commutative

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

## Associative

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

## Distributive

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

## Absorption

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

## Negation

- $p \vee \neg p \equiv \mathbf{T}$
- $p \wedge \neg p \equiv \mathbf{F}$

Law of Implication

DeMorgan

Associative

Associative

Commutative

Associative

Commutative (twice)

Negation (twice)

Domination/Identity

# Logical Proofs of Equivalence/Tautology

---

- Not smaller than truth tables when there are only a few propositional variables...
- ...but usually *much shorter* than truth table proofs when there are many propositional variables
- A big advantage will be that we can extend them to a more in-depth understanding of logic for which truth tables don't apply.