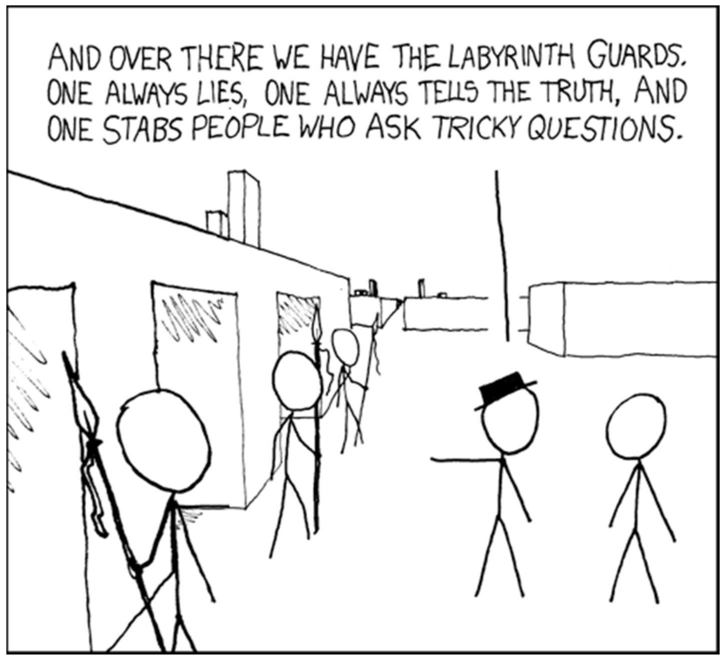
CSE 311: Foundations of Computing

Lecture 3: Digital Circuits & Equivalence



Homework #1

- You should have received
  - An e-mail from [cse311] with information pointing you to Canvas to submit HW
  - An e-mail from UW Canvas with a notification about the homework assignment. Click on "Assignments" to see all the questions

If you haven't received one, send e-mail to cse311-staff@cs.washington.edu

Last class: Logical Equivalence $A \equiv B$

$A \equiv B$ is an assertion that *two propositions* A and B always have the same truth values.

$A \equiv B$ and $\underbrace{(A \leftrightarrow B)}_{\text{tautology}} \equiv T$ have the same meaning.

$p \wedge q \equiv q \wedge p$

p	q	$p \wedge q$	$q \wedge p$	$(p \wedge q) \leftrightarrow (q \wedge p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

$p \wedge q \not\equiv q \vee p$

When p=T and q=F, $p \wedge q$ is false, but $q \vee p$ is true

Last class: De Morgan's Laws

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Last class: Equivalences Related to Implication

Law of Implication

$$p \to q \equiv \neg p \vee q$$

Contrapositive

$$p \to q \equiv \neg q \to \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Last class: Properties of Logical Connectives

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

One more easy equivalence

Double Negation

$$p \equiv \neg\neg p$$

p	$\neg p$	$\neg\neg p$	$p \leftrightarrow \neg\neg p$
T	F	T	T
F	T	F	T

Last class: Digital Circuits

Computing With Logic

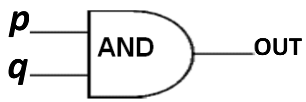
– T corresponds to 1 or "high" voltage
– F corresponds to 0 or "low" voltage

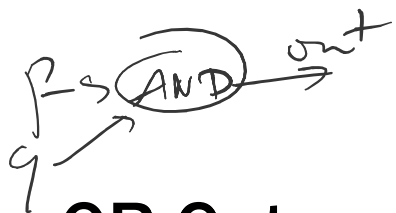
Gates

– Take inputs and produce outputs (functions)
– Several kinds of gates
– Correspond to propositional connectives (most of them)

Last class: AND, OR, NOT Gates

AND Gate

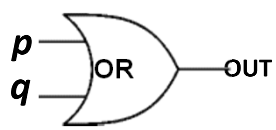


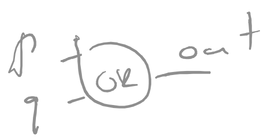


p	q	OUT
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

OR Gate

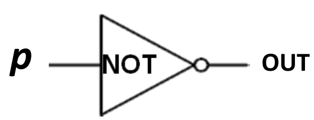




p	q	OUT
1	1	1
1	0	1
0	1	1
0	0	0

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

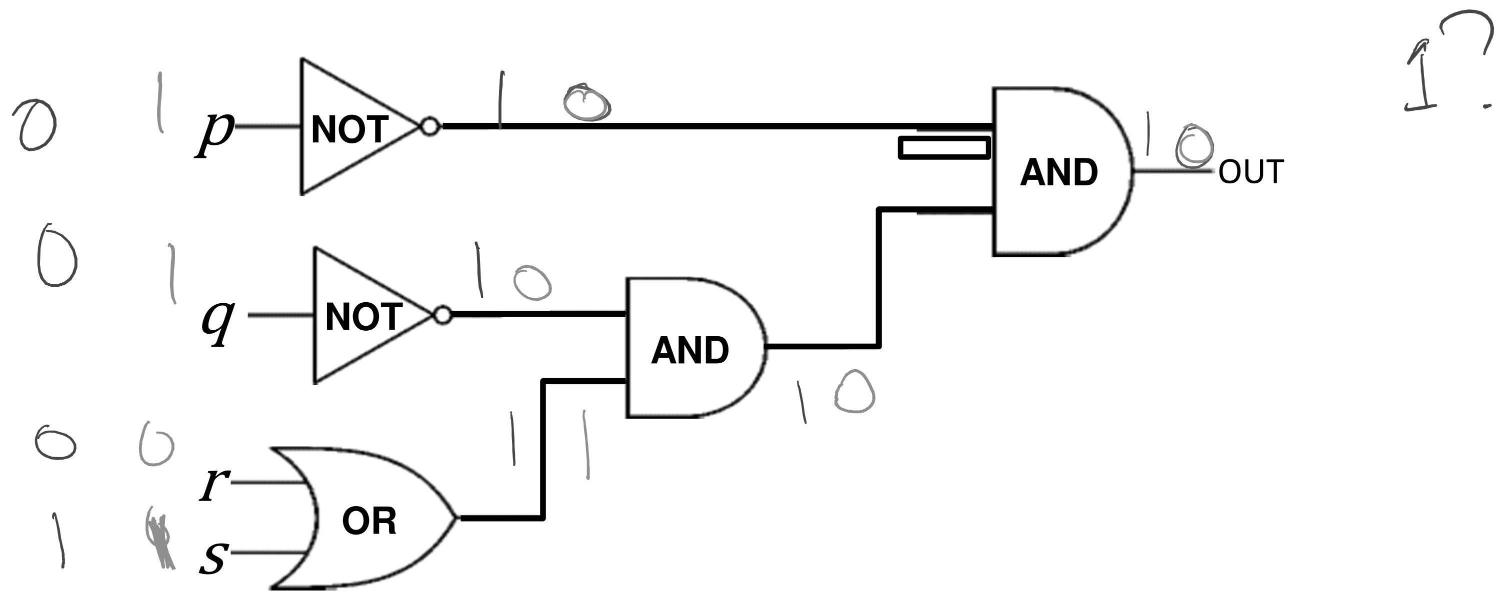
NOT Gate



p	OUT
1	0
0	1

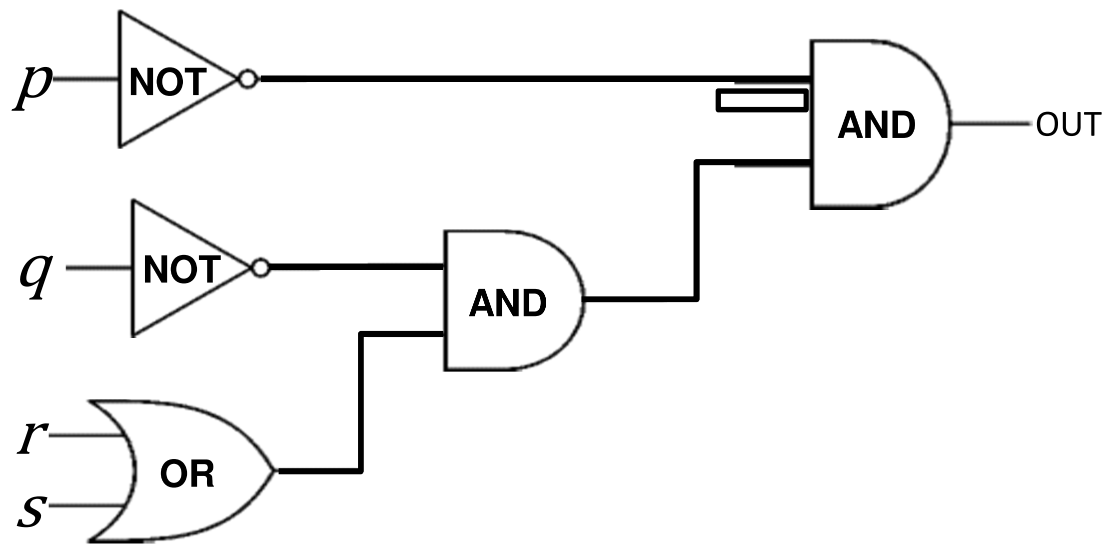
p	$\neg p$
T	F
F	T

Combinational Logic Circuits



Values get sent along wires connecting gates

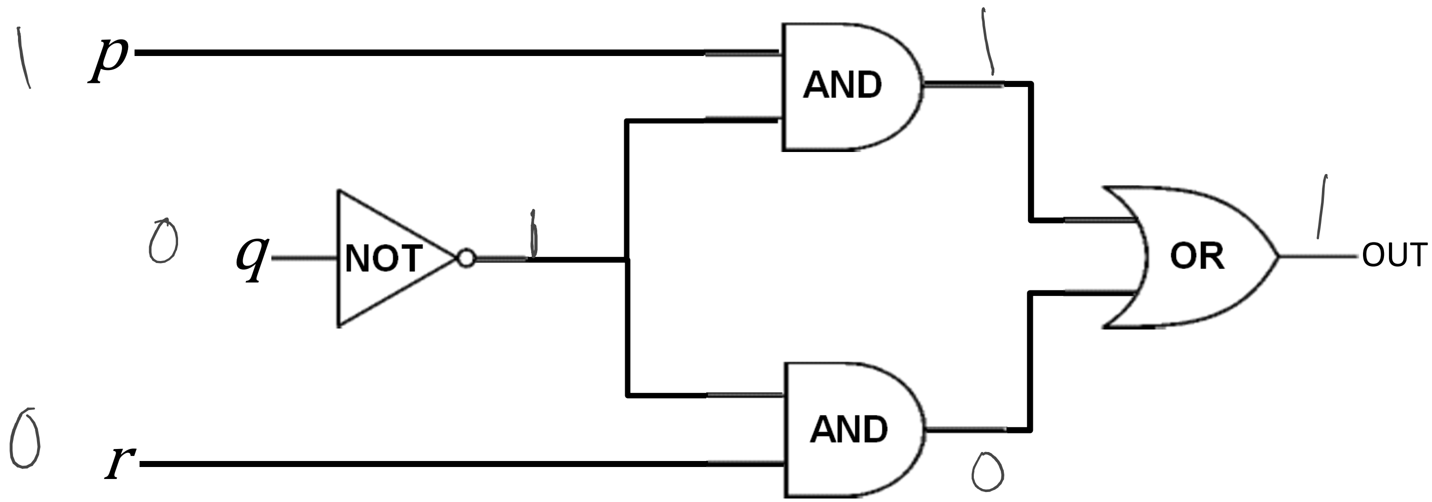
Combinational Logic Circuits



Values get sent along wires connecting gates

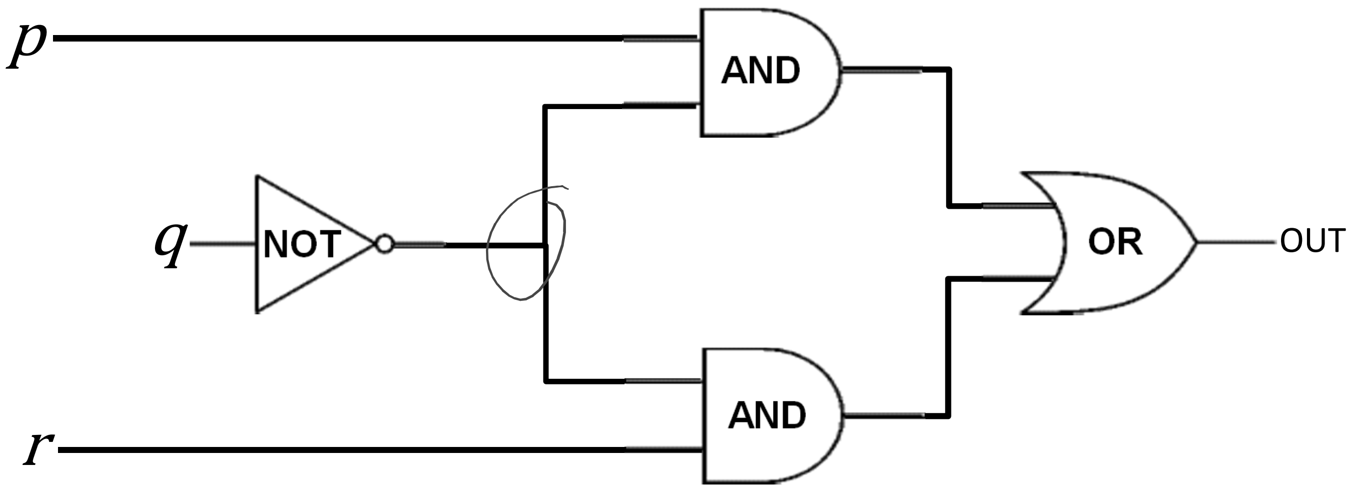
$$\neg p \wedge (\neg q \wedge (r \vee s))$$

Combinational Logic Circuits



Wires can send one value to multiple gates!

Combinational Logic Circuits



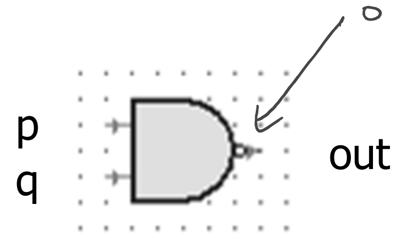
Wires can send one value to multiple gates!

$$(p \land \neg q\ ) \lor (\neg q \land r)$$

Other Useful Gates

NAND

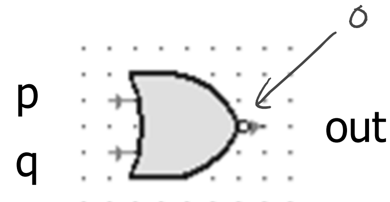
$\neg(p \land q)$



p	q	out
0	0	1
0	1	1
1	0	1
1	1	0

NOR

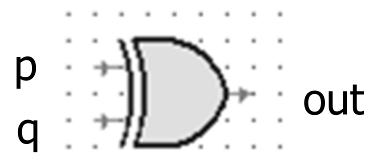
$\neg(p \lor q)$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	0

XOR

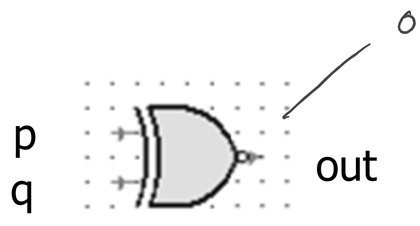
$p \oplus q$



p	q	out
0	0	0
0	1	1
1	0	1
1	1	0

XNOR

$p \leftrightarrow q$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	1

Understanding logic and circuits

When do two logic formulas mean the same thing?

When do two circuits compute the same function?

What logical properties can we infer from other ones?

Basic rules of reasoning and logic

- Allow manipulation of logical formulas
  - Simplification
  - Testing for equivalence
- Applications
  - Query optimization
  - Search optimization and caching
  - Artificial Intelligence
  - Program verification

Computing Equivalence

Given two propositions, can we write an algorithm to determine if they are equivalent?

Compute a Truth tables

What is the runtime of our algorithm?

n propositional variables

2^n rows $\geq 2^n$ time

Computing Equivalence

Given two propositions, can we write an algorithm to determine if they are equivalent?

Yes! Generate the truth tables for both propositions and check if they are the same for every entry.

What is the runtime of our algorithm?

Every atomic proposition has two possibilities (T, F). If there are n atomic propositions, there are 2^n rows in the truth table.

Another approach: Logical Proofs

To show A is equivalent to B

- Apply a series of logical equivalences to sub-expressions to convert A to B

To show A is a tautology

- Apply a series of logical equivalences to sub-expressions to convert A to T

Another approach: Logical Proofs

To show A is equivalent to B

- Apply a series of logical equivalences to sub-expressions to convert A to B

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general proof looks like:

$$
\begin{aligned}
p \vee (p \wedge p) &\equiv (\qquad\qquad\qquad\qquad\qquad) \\
&\equiv p
\end{aligned}
$$

Another approach: Logical Proofs

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$

- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general proof looks like:

$$p \vee (p \wedge p) \equiv (\; p \vee p \;) \quad \text{Idempotent}$$
$$\equiv p \quad \text{Idempotent}$$

Logical Proofs

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
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- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \to q \equiv \neg p \vee q$$

Contrapositive

$$p \to q \equiv \neg q \to \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general proof looks like:

$p \vee (p \wedge p)$	$\equiv (\quad p \vee p \quad)$	Idempotent
	$\equiv p$	Idempotent

Logical Proofs

To show A is a tautology

– Apply a series of logical equivalences to sub-expressions to convert A to **T**

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general proof looks like:

$$\begin{aligned} \neg p \vee (p \vee p) &\equiv (\qquad\qquad\qquad\qquad) \\ &\equiv \mathbf{T} \end{aligned}$$

Logical Proofs

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- Commutative
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- Associative
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- Negation
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general proof looks like:

$$\neg p \vee (p \vee p) \equiv (\ \neg p \vee p\ ) \quad \text{Idempotent}$$
$$\equiv \mathrm{T} \quad \text{Negation}$$

Logical Proofs

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$

- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general proof looks like:

$\neg\neg p \wedge p$

$$\neg p \vee (p \vee p) \equiv (\quad \neg p \vee p \quad) \quad \text{Idempotent}$$
$$\equiv \mathrm{T} \quad \text{Negation}$$

Prove these propositions are equivalent: Option 1

Prove: $p \wedge (p \to q) \equiv p \wedge q$

Make a Truth Table and show:

$$(p \wedge (p \to q)) \leftrightarrow (p \wedge q) \equiv \mathbf{T}$$

p	q	$p \to q$	$(p \wedge (p \to q))$	$p \wedge q$	$(p \wedge (p \to q)) \leftrightarrow (p \wedge q)$
T	T	T	T	T	T
T	F	F	F	F	T
F	T	T	F	F	T
F	F	T	F	F	T

Prove these propositions are equivalent: Option 2

Prove: $p \wedge (p \to q) \equiv p \wedge q$

$p \wedge (p \to q) \equiv p \wedge (\neg p \vee q)$		Law of Implication
$\equiv (p \wedge \neg p) \vee (p \wedge q)$		Distributive
$\equiv F \vee (p \wedge q)$		Negation
$\equiv (p \wedge q) \vee F$		Commutative
$\equiv p \wedge q$		Identity

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \to q \equiv \neg p \vee q$$

Contrapositive

$$p \to q \equiv \neg q \to \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Double Negation

$$p \equiv \neg\neg p$$

Prove these propositions are equivalent: Option 2

Prove: $p \wedge (p \rightarrow q) \equiv p \wedge q$

$p \wedge (p \rightarrow q) \equiv p \wedge (\neg p \vee q)$	**Law of Implication**
$\equiv (p \wedge \neg p) \vee (p \wedge q)$	**Distributive**
$\equiv \mathrm{F} \vee (p \wedge q)$	**Negation**
$\equiv (p \wedge q) \vee \mathrm{F}$	**Commutative**
$\equiv p \wedge q$	**Identity**

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- Commutative
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- Associative
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- Negation
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Prove this is a Tautology: Option 1

$$(p \land q) \to (q \lor p)$$

Make a Truth Table and show:

$$(p \land q) \to (q \lor p) \equiv \mathbf{T}$$

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \land q}$	$\boldsymbol{q \lor p}$	$\boldsymbol{(p \land q) \to (q \lor p)}$
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**T**
F	**T**	**F**	**T**	**T**
F	**F**	**F**	**F**	**T**

Prove this is a Tautology: Option 2

$$(p \wedge q) \rightarrow (q \vee p)$$

Use a series of equivalences like so:

$$\begin{aligned}
(p \wedge q) \rightarrow (q \vee p) &\equiv \neg(p \wedge q) \vee (q \vee p) && \text{Law of Implication} \\
&\equiv (\neg p \vee \neg q) \vee (q \vee p) && \text{De Morgan's} \\
&\equiv \neg p \vee (\neg q \vee (q \vee p)) && \text{Associative Law} \\
&\equiv \neg p \vee ((\neg q \vee q) \vee p) && \text{Associat} \\
&\equiv \neg p \vee (p \vee (\neg q \vee q)) && \text{Commutative Law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{Assoc.} \\
&\equiv (p \vee \neg p) \vee (q \vee \neg q) && \text{Commutative (twice)} \\
&\equiv T \vee T && \text{Neg. (twice)} \\
&\equiv \mathbf{T}
\end{aligned}$$

Associative
- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Distributive
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Absorption
- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

Negation
- $p \vee \neg p \equiv \mathrm{T}$
- $p \wedge \neg p \equiv \mathrm{F}$

Identity
- $p \wedge \mathrm{T} \equiv p$
- $p \vee \mathrm{F} \equiv p$

Domination
- $p \vee \mathrm{T} \equiv \mathrm{T}$
- $p \wedge \mathrm{F} \equiv \mathrm{F}$

Idempotent
- $p \vee p \equiv p$
- $p \wedge p \equiv p$

Commutative
- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

Prove this is a Tautology: Option 2

$$(p \wedge q) \rightarrow (q \vee p)$$

Use a series of equivalences like so:

$(p \wedge q) \rightarrow (q \vee p) \equiv \neg(p \wedge q) \vee (q \vee p)$	**Law of Implication**
$\equiv (\neg p \vee \neg q) \vee (q \vee p)$	**DeMorgan**
$\equiv \neg p \vee (\neg q \vee (q \vee p))$	**Associative**
$\equiv \neg p \vee ((\neg q \vee q) \vee p)$	**Associative**
$\equiv \neg p \vee (p \vee (\neg q \vee q))$	**Commutative**
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	**Associative**
$\equiv (p \vee \neg p) \vee (q \vee \neg q)$	**Commutative (twice)**
$\equiv \mathbf{T} \vee \mathbf{T}$	**Negation (twice)**
$\equiv \mathbf{T}$	**Domination/Identity**

Associative
- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Distributive
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Absorption
- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

Negation
- $p \vee \neg p \equiv \mathrm{T}$
- $p \wedge \neg p \equiv \mathrm{F}$

Identity
- $p \wedge \mathrm{T} \equiv p$
- $p \vee \mathrm{F} \equiv p$

Domination
- $p \vee \mathrm{T} \equiv \mathrm{T}$
- $p \wedge \mathrm{F} \equiv \mathrm{F}$

Idempotent
- $p \vee p \equiv p$
- $p \wedge p \equiv p$

Commutative
- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

Logical Proofs of Equivalence/Tautology

- Not smaller than truth tables when there are only a few propositional variables…

- …but usually *much shorter* than truth table proofs when there are many propositional variables

- A big advantage will be that we can extend them to a more in-depth understanding of logic for which truth tables don't apply.